PAPERS on Economics & Evolution

1117

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The *Papers on Economics and Evolution* are edited by the Evolutionary Economics Group, MPI Jena. For editorial correspondence, please contact: evopapers@econ.mpg.de

ISSN 1430-4716

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Attractive evolutionary equilibria*

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December 7, 2011

Abstract

We present attractiveness, a refinement criterion for evolutionary equilibria. Equilibria surviving this criterion are robust to small perturbations of the underlying payoff system or the dynamics at hand. Furthermore, certain attractive equilibria are equivalent to others for certain evolutionary dynamics. For instance, each attractive evolutionarily stable strategy is an attractive evolutionarily stable equilibrium for certain barycentric ray-projection dynamics, and vice versa.
Key words: attractive evolutionary equilibria, evolutionary dynamics, evolutionary, dynamic & structural stability.
JEL-Codes: C62; C72; C73.

1 Introduction

The evolutionarily stable strategy (*ESS*) of Maynard Smith & Price [1973] is probably the best known concept from evolutionary game theory, rivaled only by the replicator dynamics of Taylor & Jonker [1978]. The state $y \in S^n$ is evolutionarily stable strategy if and only if a nonempty open neighborhood $U \subset S^n$ exists such that $U \ni y$, and $x \in U \backslash \{y\}$ implies

$$(y - x) \cdot f(x) > 0. \tag{1}$$

Here, y and x are vectors of population shares or alternatively interpreted, mixed strategies, S^n denotes the n-dimensional unit simplex. The function f is a relative fitness function (cf., Joosten [1996]), called an excess payoff function elsewhere (e.g., Sandholm [2005]). Such a (vector)function attributes to every subgroup in the population its fitness relative to the population share weighted average fitness.

*Reinoud Joosten thanks Dorothea Herreiner for inspiring questions in Stony Brook on another paper. Attractive has at least two meanings, 'pulling in' as well as 'appealing'. Both meanings apply, at least subjectively.

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The *ESS*-concept is meant to capture that if a population in equilibrium is invaded by any sufficiently small group, the system will return to the original equilibrium state. This suggests a close relationship with stability as used in the analysis of dynamic systems, but the *ESS* has been concipated purely statically.[1] Early efforts of linking the *ESS* with a dynamic system for which it coincides with an asymptotically stable fixed point are Taylor & Jonker [1978], Zeeman [1981] and Hofbauer *et al.* [1979].

Certain other asymptotically stable fixed points of evolutionary dynamics have been proposed. For instance, the state $y \in S^n$ is evolutionarily stable equilibrium (*ESE,* Joosten [1996]) if and only if an open neighborhood $U \subset S^n$ exists such that $U \ni y$, and $x \in U \backslash \{y\}$ implies

$$(y - x) \cdot h(x) > 0, \tag{2}$$

where $h : S^n \to \mathcal{O}^{n+1} = \{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i = 0\}$. Here, h represents the evolutionary dynamics at hand. Slightly more formally, we have a system of $(n + 1)$ autonomous differential equations

$$\frac{dx}{dt} = h(x) \text{ for all } x \in S^n.$$

Here, $\frac{dx}{dt}$ denotes the continuous-time change of composition of the population, or alternatively the mixed strategy.[2] It can be easily proven that (2) implies that all trajectories sufficiently near y move towards it monotonically, i.e., the Euclidean distance to the equilibrium decreases steadily. This, of course, implies that y is an asymptotically stable fixed point.

We are interested in stability in a broader sense than usually considered in evolutionary game theory. Given evolutionary equilibrium $y \in S^n$ for which $f(y) = h(y) = 0^{n+1}$ satisfying some properties P leading to consequences C, we can imagine perturbations to the tuple (y, f, h, P). We want the perturbed system to be qualitatively similar to the original. For instance, if an evolutionarily stable equilibrium y is perturbed slightly to $x_0 \in U$, then for unperturbed f, h the property P given by (2) still holds and the consequence C is that $\{x_t\}_{t \geq 0} \overset{t \to \infty}{\to} y$ under the dynamics. This is the familiar question of dynamic stability, of course, but what can be said about C for perturbations of f and\or h? What about 'perturbations' of P?

We do not intend to examine all possible perturbations. We restrict attention to the following interesting questions. Can we formulate refinements of evolutionary equilibrium concepts that give us back[3] structural stability

[1] Yet the profession remains doggedly faithful to the concept. One of us vented his amazement on this elsewhere (Joosten [2010]).

[2] The dynamics h should be connected to the relative fitness function and the different classes do so in different manners with various interesting motivations, e.g., Sandholm [2005,2010], Schlag [1998,1999], Hofbauer & Sigmund [1998]. Section 3 treats dynamics.

[3] Taylor & Jonker [1978] and Zeeman [1981] use methods implying dynamic and structural stability. Proofs using Lyapunov's second method may allow more general results on stability of fixed points, but razor sharp formulations reduce the robustness of results.

of some kind? How damaging to our predictions is a slight error in assessing f or h? How robust are our consequences with respect to imperfect descriptions of the dynamics or the underlying system? Can we formulate conditions such that *ESS*-stability and other types of evolutionary stability such as *ESE*-stability concur?[4] To answer these few questions of this research agenda is already bound to be too ambitious for one paper, unless we limit the range of games, dynamics or alternatively the type of equilibria for which the property is to hold (see also e.g., Sandholm [2010a]).

Here, we focus on the latter, we examine the following concept to be used as a refinement of evolutionary equilibria. For given functions $z^1 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $z^2 : S^n \to \mathbb{R}^{n+1}$ and $y \in S^n$ we say that y is **attractive** with respect to $z = \left(z^1, z^2\right)$ iff (i) $z^1(y) \cdot z^2(y) = 0$ and (ii) an $\varepsilon \in (0, 1)$ and an open neighborhood $U \subset S^n$ exist such that $U \ni y$, and $x \in U\backslash\{y\}$ implies

$$\frac{z^1(x) \cdot z^2(x)}{||z^1(x)|| \cdot ||z^2(x)||} > \varepsilon.$$

A state is weakly attractive if this inequality holds for $\varepsilon = 0$. Recall that the expression before the inequality sign is the cosine of the angle between the two vectors involved. We connect z to the mathematics defining an equilibrium whenever possible; for instance, an *ESS* $y \in S^n$ is attractive if in the above $z^1(x) = (y - x)$ and $z^2(x) = f(x)$. Attractiveness induces a refinement of the *ESS* concept, the weaker form coincides with it.

Our results show that attractive evolutionary equilibria preserve their defining properties for a series of perturbations of the payoff system or the dynamics. So, slight mis-specifications of either are harmless with respect to conclusions regarding the dynamic stability of the equilibrium at hand.

More interesting, by showing equivalence of attractive evolutionary equilibria of different origins, our conclusions turn out to be robust against slight discrepancies in specifications of the equilibrium concept at hand as well. For instance, each attractive evolutionarily stable strategy is an attractive evolutionarily stable equilibrium under a large subclass of the barycentric ray-projection dynamics, and vice versa. We also present first results on equivalences between attractive evolutionarily stable strategies and attractive truly evolutionarily stable strategies on the one hand, and attractive generalized evolutionarily stable equilibria on the other.

In the next section, we present equilibrium concepts in evolutionary game theory and then introduce their 'attractive' variants; in Section 3 we define evolutionary dynamics to be used. In Section 4 we show that attractive equilibria are robust to perturbations of the underlying payoff system or the dynamics. Section 5 is devoted to showing equivalences between attractive equilibria. Section 6 discusses generalizations and further research, Section 7 concludes. All proofs can be found in the Appendix.

[4] Relating to the issue of perturbing property P.

2 Evolutionary equilibria and attractiveness

Let $x \in S^n$ denote a vector of population shares for a population with $n+1$ distinguishable, interacting subgroups. Here, S^n is the n-dimensional unit simplex, i.e., the set of all non-negative $n+1$-dimensional vectors with components adding up to unity. The interaction of the subgroups has consequences on their respective abilities to reproduce, and 'fitness' may be seen as a measure of this ability to reproduce. As behavior of each subgroup is assumed essentially predetermined, fitness depends only on the state of the system, i.e., the composition of the population.

Let $F : S^n \to \mathbb{R}^{n+1}$ be a **fitness function**, i.e., a continuous function attributing to every subgroup its fitness at each state $x \in S^n$. Then, the **relative fitness function** $f : S^n \to \mathbb{R}^{n+1}$ is given by:

$$f_i(x) = F_i(x) - \sum_{j=1}^{n+1} x_j F_j(x), \text{ for all } i \in I^{n+1} = \{1, ..., n+1\}, \ x \in S^n.$$

So, a relative fitness function attributes to each subgroup the difference between its fitness and the population share weighted average fitness taken over all subgroups.

We already gave the definitions of the evolutionarily stable strategy (*ESS*) and the evolutionarily stable equilibrium (*ESE*) in the introduction. The state $y \in S^n$ is a **saturated equilibrium** if $f(y) \leq \mathbf{0}^{n+1}$, a **fixed point** if $h(y) = \mathbf{0}^{n+1}$; a fixed point y is **(asymptotically) stable** if, for any neighborhood $U \subset S^n$ of y, there exists an open neighborhood $V \subset U$ of y such that any trajectory starting in V remains in U (and converges to y). At a saturated equilibrium all subgroups with below average fitness have population share equal to zero. So, rather than 'survival of the fittest', we have 'extinction of the less fit'.

A saturated equilibrium $y \in S^n$ is called **strict** if $f_j(y) = 0$ for precisely one $j \in I^{n+1}$ in an open neighborhood $U \subset S^n$ of y. Every strict saturated equilibrium is a vertex of the unit simplex. The saturated equilibrium is due to Hofbauer & Sigmund [1988], the strict version to Joosten [1996].

The fixed point $y \in S^n$ is a **generalized evolutionarily stable state** (*GESS*, Joosten [1996]) if and only if there exists an open neighborhood $U \subset S^n$ of y such that (1) holds. The *GESS* generalizes the *ESS* of Maynard Smith & Price [1973] in order to deal with *arbitrary* relative fitness functions[5]. If the fitness function is given by $F(x) = Ax$ for some square matrix A, every (strict) saturated equilibrium coincides with a (strict) Nash equilibrium of the evolutionary bi-matrix game $(A, A^\top)$; moreover, the *GESS* and *ESS* coincide.

[5] A relative fitness function is characterized by continuity and complementarity, i.e., $x \cdot f(x) = 0$ for all $x \in S^n$.

Another concept also inspired by *ESS* avoids the traditional mistake of defining a static evolutionary equilibrium concept. The fixed point $y \in S^n$ is a **truly evolutionarily stable state** iff a nonempty open neighborhood $U \subset S^n(C(y))$ containing y exists such that

$$\sum_{i \in C(y)} (y_i - x_i) \frac{h_i(x)}{x_i} - \sum_{i \notin C(y)} h_i(x) > 0. \tag{3}$$

The *TESS* is due to Joosten [2009] where it is shown that the condition above guarantees asymptotical stability of the equilibrium.

The following concept from Joosten [2009] generalizes the idea behind the *ESE*. Let $d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a distance function, and $V : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be differentiable, homothetic with d. Then, $y \in S^n$ is a **generalized evolutionarily stable equilibrium** if and only if a nonempty open neighborhood $U \subseteq S^n$ containing y, exists such that for all $x \in U \backslash \{y\}$ it holds that $[V(x,y) - V(y,y)] \cdot \dot{V}(x,y) < 0$, where $\dot{V}(x,y) = \sum_{i=1}^{n+1} \frac{\partial V}{\partial x_i} h_i(x)$. For a *GESE,* each trajectory sufficiently nearby converges such that the distance to it decreases monotonically under *at least one metric.*

2.1 Attractive evolutionary equilibria

For given functions $z^1 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $z^2 : S^n \to \mathbb{R}^{n+1}$ and $y \in S^n$ we say that y is **attractive** with respect to $z = (z^1, z^2)$ iff (i) $z^1(y) \cdot z^2(y) = 0$ and (ii) an $\varepsilon \in (0,1]$ and an open neighborhood $U \subset S^n$ exist such that $U \ni y$, and $x \in U \backslash \{y\}$ implies

$$\frac{z^1(x) \cdot z^2(x)}{||z^1(x)|| \cdot ||z^2(x)||} > \varepsilon. \tag{A}$$

A state $y \in S^n$ is **weakly attractive** for given z if the inequality (A) holds for $\varepsilon = 0$. The angle between $z^1(x)$ and $z^2(x)$, $x \neq y$, is acute, and bounded away from 90 degrees.

Before introducing the attractive variants of equilibrium concepts treated in the previous section, we need just another notation. Let for a *GESE* y, the function V denote the one mentioned in the definition. Clearly, a function W exists such that $W(x,y) = -|V(x,y) - V(y,y)|$ for all $x, y \in S^n$. So, the above implies $W(x,y) \leq 0$, $W(y,y) = 0$ and $\sum_{i=1}^{n+1} \frac{\partial W}{\partial x_i} h_i(x) > 0$.

Now, we are ready to give the attractive variants of four evolutionary equilibrium concepts mentioned in the preceding sections. Let $y \in S^n$, and let $U \subset S^n$ be a nonempty open neighborhood of y containing it, then

- y is an **attractive (G)ESS** iff (A) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = (y - x), z^2(x) = f(x)$,
- y is an **attractive ESE** iff (A) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = (y - x)$, $z^2(x) = h(x)$,

- y is an **attractive TESS** iff (A) holds for all $x \in U \backslash \{y\}$, with $z_i^1(x) = \frac{y_i - x_i}{x_i}$ for all $i \in I^{n+1}$, $z^2(x) = h(x)$,
- y is an **attractive GESE** iff (A) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = DW(x) \equiv \left[\frac{\partial W}{\partial x_1}, ..., \frac{\partial W}{\partial x_{n+1}}\right]$, $z^2(x) = h(x)$.

3 Evolutionary dynamics

In the sequel, we assume that there exists a given function $h : S^n \to \mathbb{R}^{n+1}$ satisfying $\sum_{j=1}^{n+1} h_j(x) = 0$ for all $x \in S^n$. Consider this system of $n + 1$ autonomous differential equations:

$$\dot{x} = \frac{dx}{dt} = h(x) \text{ for all } x \in S^n, \tag{4}$$

where $\frac{dx}{dt}$ denotes the continuous-time changes of the vector $x \in S^n$. A **trajectory** under the dynamics h is a solution, $\{x(t)\}_{t \geq 0}$, to $x(0) = x_0 \in S^n$ and Equation (4) for all $t \geq 0$. We refrain from placing too many mathematical restrictions on h at this point, we do require existence and uniqueness of trajectories. Continuity of h implies existence, and Lipschitz continuity or differentiability implies uniqueness. We refer to Perko [1991] as an excellent textbook on differential equations and dynamics.

The evolution of the composition of the population is represented by system (4). To make sense in an evolutionary framework further restrictions on the system are required. The function h is therefore assumed to be connected to the relative fitness function f in one of the many ways proposed in the literature, cf., e.g., Nachbar [1990], Friedman [1991], Swinkels [1993], Joosten [1996], Ritzberger & Weibull [1995].

For so-called **sign-compatible** evolutionary dynamics, the change in population share of each subgroup with positive population share corresponds in sign with its relative fitness; for **weakly sign-compatible** evolutionary dynamics, at least one subgroup with positive relative fitness grows.[6] Dynamics are **one-sided sign-compatible** if one of the two cases hold: (i) all subgroups having nonnegative relative fitness grow, or (ii) all non-extinct subgroups having nonpositive relative fitness shrink. Alternatives were defined by Friedman [1991]: dynamics are **weakly compatible** if $f(x) \cdot h(x) \geq 0$ for all $x \in S^n$ (with strict inequality if x is not an equilibrium), **order compatible** if $f_i(x) < f_j(x)$ implies $h_i(x) < h_j(x)$ for interior states.

Let the following functions from the interior of the n-dimensional unit simplex to $\mathcal{O}^{n+1}$, be componentwise, i.e., for all $i \in I^{n+1}$, given by:

$h_i^{BN}(x) = \max\{0, f_i(x)\} - x_i \cdot \sum_{j=1}^{n+1} \max\{0, f_j(x)\}$;

[6] Both classes due to Joosten [1996].

$$
\begin{aligned}
&h_i^{BR}(x) = \begin{cases} 1 - x_i & \text{if } i = j^* \in \{k \in I^{n+1} |\ f_k(x) = \max_{h \in I^{n+1}} f_h(x)\}, \\ -x_i & \text{otherwise.} \end{cases}; \\
&h_i^{REP}(x) = x_i f_i(x); \\
&h_i^{q-REP}(x) = x_i^q \left[f_i(x) - \frac{\sum_{j=1}^{n+1} x_j^q f_j(x)}{\sum_{j=1}^{n+1} x_j^q} \right]; \\
&h_i^{OPD}(x) = f_i(x) - \tfrac{1}{(n+1)} \textstyle\sum_{j=1}^{n+1} f_i(x); \\
&h_i^{RPD}(x) = f_i(x) - \left(\textstyle\sum_{i=1}^{n+1} f_j(x) \right) x_i; \\
&h_i^{RUN}(x) = \lim_{\varepsilon \downarrow 0} \left[[\varepsilon \cdot f_i(x) + x_i]_+ - \left(\textstyle\sum_{j=1}^{n+1} [\varepsilon \cdot f_j(x) + x_j]_+ \right) x_i \right]; \\
&h_i^{OUN}(x) = \lim_{\varepsilon \downarrow 0} \left[[\varepsilon \cdot f_i(x) + x_i]_+ - x_i + \tfrac{1}{n+1} - \left(\frac{\sum_{j=1}^{n+1} [\varepsilon \cdot f_j(x) + x_j]_+}{n+1} \right) \right]; \\
&h_i^{L}(x) = e^{f_i(x)} - \left(\textstyle\sum_{j=1}^{n+1} e^{f_j(x)} \right) x_i; \\
&h_i^{WL}(x) = x_i \left[e^{f_i(x)} - \left(\textstyle\sum_{j=1}^{n+1} x_j e^{f_j(x)} \right) \right].
\end{aligned}
$$

Above, j^* is always uniquely determined, $[y]_+ = \max\{0, y\}$, whereas superscripts *BN*, *BR*, *REP*, *q-REP*, *OPD*, *RPD*, *RUN*, *OUN*, *L* and *WL* refer to the dynamics of Brown & Von-Neumann [1950], the best-response dynamics of Gilboa & Matsui [1991] and Matsui [1992], the replicator dynamics, the q-deformed replicator dynamics (cf., Harper [2010]), the orthogonal-projection dynamics (Lahkar & Sandholm [2008]), the ray-projection dynamics (Joosten & Roorda [2011]), the generalized ray-projection and the generalized orthogonal-projection of the dynamics of Nikaidô & Uzawa [1960] (cf., Joosten & Roorda [2011]), the logit dynamics of Fudenberg & Levine [1998] and the (weighted logit, *our name*) dynamics of Björnerstedt & Weibull [1996] respectively. The q-deformed replicator dynamics for $q \in [0, 1]$ have two prominent members, the replicator dynamics ($q = 1$) and the orthogonal projection dynamics ($q = 0$).

We now focus on a variant of dynamics analyzed by Hofbauer & Sigmund [1998] and independently Hopkins [1999]. Let the 'adaptive' dynamics $h^A : S^n \to \mathcal{O}^{n+1}$ on the interior of the unit simplex be determined by

$$h^A(x) = A(x)f(x) \text{ for all } x \in int\ S^n, \tag{ADAPT}$$

where $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a continuous function attributing to every $(n+1)$-vector a symmetric, strictly positive definite $(n+1) \times (n+1)$-matrix, i.e., $yA(x)y \geq 0$ for all x, y and $yA(x)y = 0$ iff $y = 0^{n+1}$.

The dynamics should fulfill certain boundary conditions to be 'admissible' as evolutionary dynamics, but for the present purposes the above will suffice. It is easy to confirm that h^A belongs to the set of weakly compatible dynamics. The matrix function A can be regarded as a rotation operator, transforming every relative fitness vector into one pointing in the unit simplex under an acute angle with the original.

3.1 Barycentric ray-projection dynamics

We now define a class unifying the orthogonal-projection dynamics of Lahkar & Sandholm [2008] and the ray-projection dynamics of Joosten & Roorda [2011]. Let $\alpha \leq 0$, then the **α-barycentric ray-projection dynamics** of $f : int\ S^n \to \mathbb{R}^{n+1}$ are given by

$$h^{\alpha}(x) = f(x) + \frac{\sum_{i=1}^{n+1} f_i(x)}{1-(n+1)\alpha}(\alpha \cdot 1^{n+1} - x).$$

The interpretation is that $f(x)$ is projected unto the n-dimensional unit simplex on a ray leading from x to $\alpha \cdot 1^{n+1}$. Barycentric ray-projection dynamics (with finite but possibly very negative α) are not order compatible, whereas it is easy to see that the *OPD* are. Observe that

$$h^{0^{n+1}}(x) = f(x) + \frac{\sum_{i=1}^{n+1} f_i(x)}{1-\sum_{i=1}^{n+1} 0}\left(0^{n+1} - x\right) = f(x) - \left(\sum_{i=1}^{n+1} f_i(x)\right) x.$$

Moreover, if $a = \alpha \cdot 1^{n+1}$, then

$$\begin{aligned} h^{-\infty^{n+1}}(x) &\equiv \lim_{\alpha \to -\infty} h^{a}(x) = \lim_{\alpha \to -\infty} \left[f(x) + \frac{\sum_{i=1}^{n+1} f_i(x)}{1-(n+1)\,\alpha}\left(\alpha \cdot 1^{n+1} - x\right)\right] \\ &= f(x) - \left(\sum_{i=1}^{n+1} f_i(x)\right) \frac{1}{(n+1)} \cdot 1^{n+1}. \end{aligned}$$

The former type of projection dynamics are the ray-projection dynamics, the latter the orthogonal projection dynamics. The ensuing result sheds light on the positioning of the barycentric ray-projection dynamics.

Lemma 1 *Barycentric ray-projection dynamics satisfy weak compatibility and weak sign-compatibility for finite* $\alpha \leq 0$.

We have summarized several connections between notions defined here and the previous sections in two figures. Figure 1 deals with equilibria under different dynamics presented. (S)*SAT*, (G)*ESE*, (G)*ESS*, *TESS*, (A)*SFP* and *FP* denote the sets of (strictly) saturated equilibria, (generalized) evolutionarily stable equilibria, (generalized) evolutionarily stable states, truly evolutionary stable states, (asymptotically) stable fixed points and fixed points respectively. Figure 2 visualizes relations between classes of evolutionary dynamics.

4 Perturbations of payoffs or dynamics

Given $z^2 : S^n \to \mathbb{R}^{n+1}$, $\theta \in (0,1)$ and $\varepsilon > 0$, let $Z^{\theta}(z^2, \varepsilon)$ be the set of **continuous** functions perturbations of z^2 given by

$$Z^{\theta}(z^2, \varepsilon) = \left\{ z : S^n \to \mathbb{R}^{n+1} | \frac{z^2(x) \cdot z(x)}{||z^2(x)|| \cdot ||z(x)||} \geq \sqrt{1-(\theta\varepsilon)^2} \right\}.$$

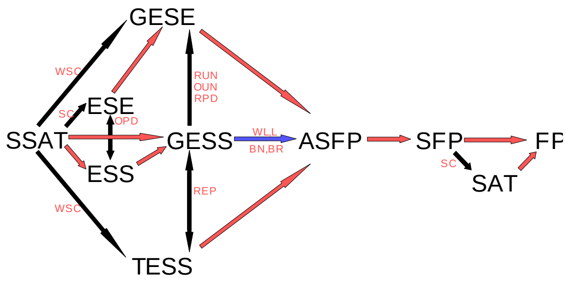

Figure 1: Arrows indicate inclusions; red indicates a general one, black for special (classes of) dynamics, blue under special conditions. Abbreviations coincide with those in the superscripts in Section 3; $(W)SC$ denotes (weakly) sign-compatible dynamics.

Obviously, $Z(z^2, \varepsilon) = \cup_{\theta \in (0,1)} Z^\theta(z^2, \varepsilon)$ is nonempty as it must contain z^2 to be obtained as the limit for $\theta \to 0$, $\varepsilon \to 0$. The following result links the ε above to the same parameter in the definition of an attractive equilibrium and specifies a lower bound for the cosine between z^1 and a perturbation taken from the set above.

Proposition 2 *Let $y \in int\ S^n$ be an attractive (G)ESS (ESE, TESS or GESE) and let z in $Z^\theta(f, \varepsilon)$ $(Z^\theta(h, \varepsilon))$ satisfy $z(y) = 0^{n+1}$ then*

$$\frac{z^1(x) \cdot z(x)}{||z^1(x)|| \cdot ||z(x)||} > \varepsilon \left(\sqrt{1 - (\theta\varepsilon)^2} - \sqrt{\theta^2 - (\theta\varepsilon)^2} \right).$$

We see the important role of ε here, the closer ε is to unity, the more slack can be offered to the perturbations. The other parameter θ is necessary to specify the part behind the inequality sign. This result has the following convenient consequence.

Corollary *For every attractive evolutionary equilibrium concept presented, a set of sufficiently small perturbations of the payoffs or dynamics can be found such that the equilibrium is attractive under these perturbations, too.*

We conclude this (sub)section with a summarizing remark. Attractive evolutionary equilibria presented here are refinements of the concepts closely associated. All *ESE*, *TESS* and *GESE* are asymptotically stable fixed points of the dynamics at hand anyway, so their attractive variants are asymptotically stable fixed points as well. For sufficiently small disturbances of the evolutionary dynamics, the dynamic stability of the attractive variants is

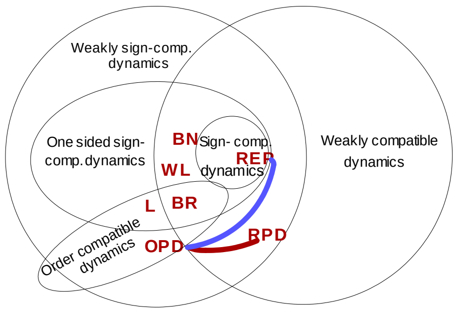

Figure 2: Relations between dynamics. The red line depicts barycentric ray-projection dynamics; the blue one q-deformed replicator dynamics, $q \in [0, 1]$.

not jeopardized. Likewise for the attractive *(G)ESS*, its defining property is not endangered for sufficiently small perturbations of the relative fitness function at hand. Hence, results on dynamic stability of the *(G)ESS* for certain dynamics still hold.

5 Equivalences between attractive equilibria

We now examine whether equivalences of certain attractive evolutionary equilibria can be shown to hold for certain dynamics. As a starting point, we focus on the attractive variants of the *(G)ESS* and the *ESE*. For this, we begin with the following equivalence for $x \in int\ S^n \backslash \{y\}$

$$(y - x) \cdot f(x) = (y - x) \cdot \left[f(x) - \left(\sum_{j=1}^{n+1} f_j(x) \right) \cdot 1^{n+1} \right] = (y - x) \cdot h^{OPD}(x).$$

So, *ESS* and *ESE* concur for the orthogonal-projection dynamics of Lahkar & Sandholm [2008]. We use this identity for the ensuing result.

Proposition 3 *For $y \in int\ S^n$, the following statements are equivalent:*

- *y is an attractive ESE under the orthogonal-projection dynamics;*
- *y is an attractive (G)ESS.*

In light of the conclusions made in the previous section, both types of attractive evolutionary equilibria keep their defining properties under a series of sufficiently small perturbations. In the following, we want to establish not individual examples of dynamics (and perturbations thereof) for which said equivalence holds, but rather a class of dynamics. Our natural ally in this endeavor remains the *OPD*, as the above shows a promising start. We generalize the above slightly to incorporate the more general barycentric projection dynamics. The next result reveals some potential in this respect.

Lemma 4 *Let* $a = \alpha \cdot 1^{n+1}$ *and* h^α *denote the* α*-barycentric ray-projection dynamics, then for* $x \in int\ S^n$:

- $\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \geq \frac{1}{2}\sqrt{2}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} - \sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha}$;
- $\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} \geq \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha}$.

This lemma clearly extends the proposition preceding it. These inequalities are important in proving equivalence of attractive *ESS*s and *ESE*s. One must make sure that $\sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha}$ is sufficiently small for the right hand sides of the inequalities to be strictly positive if the goal is to show equivalence between attractive *ESS* and attractive *ESE* for certain barycentric projection dynamics. The next result hinges on the fact that the part behind the minus sign can be made arbitrarily small.

Proposition 5 *For* $y \in int\ S^n$, α_0 *exists such that for all* $\alpha \leq \alpha_0$ *the following statements are equivalent:*

- *y is an attractive ESE for the α-barycentric ray-projection dynamics;*
- *y is an attractive (generalized) ESS.*

We now turn to showing equivalence of a similar nature for the attractive *ESS* and the attractive *TESS*. Here our natural ally must be the replicator dynamics as may be guessed from the overview in Figure 2.

Proposition 6 *For* $y \in int\ S^n$*, the following statements are equivalent:*

- *y is an attractive (generalized) ESS;*
- *y is an attractive TESS under the replicator dynamics.*

If we want to generate a broader result in the spirit of our previous result, we may search support from within the class of q-deformed replicator dynamics (cf., Figure 2).

Proposition 7 *For* $y \in int\ S^n$ *and* q *sufficiently near 1, the following statements are equivalent:*

- *y is an attractive (generalized) ESS;*
- *y is an attractive TESS under the q-deformed replicator dynamics.*

Next, we intend to construct dynamics allowing a similar result. Following Joosten & Roorda [2008] regarding generalized projection dynamics, we introduce the function g^p as a perturbation of the replicator dynamics in the following manner (componentwise and for interior states x):

$$g_i^p(x) = x_i f_i(x) - \min\left\{ f_i(x)^{2p}, p^{-2p} \right\}.$$

Here, p is a natural number sufficiently large to guarantee that near an interior equilibrium the perturbation term goes to zero quickly. It can be confirmed that near an interior equilibrium $g^p(x) \overset{p\to\infty}{\rightarrow} h^{REP}(x)$. Note that this function does not induce dynamics on S^n. The following dynamics do and they are given, componentwise and for interior states x, by

$$h_i^{\alpha,p-PR}(x) = g_i^p(x) - \frac{\alpha - x_i}{\alpha(n+1)-1}\sum_{j=1}^{n+1} g_j^p(x). \tag{PR}$$

We refer to these dynamics as the α, p-P(erturbed)R(eplicator) dynamics, because first the replicator dynamics are perturbed to dynamics not necessarily forward invariant with respect to the interior of the unit simplex and then projected back on the unit simplex along the ray $a - x$, where $a = \alpha \cdot 1^{n+1}$ as before. In the following equivalence result, the latter dynamics will play an important role.

Proposition 8 *For $y \in int\ S^n$, let $h^{\alpha,p-PR}$ given by (PR) determine the dynamics. Then, the following statements are equivalent:*

- *y is an attractive (generalized) ESS;*
- *y is an attractive TESS.*

We now focus on 'adaptive' dynamics. Let dynamics $h^A : S^n \to \mathcal{O}^{n+1}$ be determined by

$$h^A(x) = A(x)f(x) \text{ for all } x \in int\ S^n.$$

Recall that (ADAPT) implies that every matrix $A(x)$ is symmetric and strictly positive definite. It is well-established that any such matrix possesses an inverse matrix with the same properties. Let $y \in int\ S^n$ satisfy $f(y) = 0^{n+1}$ and let

$$V_y(x) \equiv (y - x) \cdot A^{-1}(y)(y - x) \text{ for all } x \in S^n.$$

Clearly, $V_y(x) = 0$ if $x = y$, and $V_y(x) > 0$ otherwise, and $\frac{dV_y(x)}{dt} = -2(y - x) \cdot A^{-1}(y)A(x)f(x)$. So, V_y can be regarded as a Lyapunov function iff an open neighborhood $U \ni y$ exists such that for all $x \in U \backslash \{y\}$

$$(y - x) \cdot A^{-1}(y)A(x)f(x) > 0. \tag{STAB}$$

Furthermore, V_y can be regarded as a metric, hence y is a generalized evolutionarily stable equilibrium. Now, we define y as an **attractive** *GESE* **w.r.t.** V_y if an open neighborhood $U \ni y$ exists, such that

$$\frac{(y-x)\cdot A^{-1}(y)h^A(x)}{||y-x||\cdot||A^{-1}(y)h^A(x)||} > \varepsilon \text{ for all } x \in U\backslash\{y\}.$$

The next result gives a welcome addition to the equivalences aimed at here.

Proposition 9 *For $y \in$ int S^n and adaptive dynamics represented by h^A satisfying (ADAPT), the following statements are equivalent.*

- *y is an attractive (generalized) ESS;*
- *y is an attractive GESE w.r.t. V_y.*

Joosten [2009] showed that monotone convergence in one metric does not necessarily hold for another. Monotone convergence for a metric means that near an equilibrium all trajectories converge to it such that the distance to the equilibrium as measured by that metric strictly decreases monotonically in time. So, y satisfying (STAB) is a *GESE* for V_y, but not necessarily for another distance function. Hence, the final result can not be included in the overview of Figure 3 in the same manner as the other equivalences.

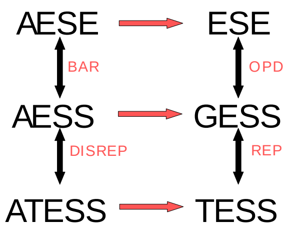

Figure 3: Overview of equivalences shown, arrows denote inclusions: red for general, black for special dynamics. BAR denotes the barycentric projection dynamics of Prop. 5, DISREP denotes disturbed replicator dynamics, containing at least the dynamics in Prop. 7 & 8.

6 Discussion

We already mentioned that it is impossible to bound away the defining equations of the four equilibrium concepts away from zero by a constant as $(y-x)$, $h(x)$, $f(x) \overset{y\to x}{\to} 0^{n+1}$. The same can be said with respect to an alternative definition of the evolutionarily stable state, i.e., $(y-x)\cdot F(x) \overset{y\to x}{\to} 0$. Here F is the fitness function instead of the relative fitness function.

It is interesting to note that the analogy of attractiveness with respect the fitness function F can not be constructed as

$$\frac{(y-x)\cdot F(x)}{||y-x||\cdot||F(x)||} \overset{x\to y}{\to} 0.$$

Observe that $F(y) = c \cdot 1^{n+1}$ for an interior equilibrium. As a result, $F(x)$ becomes 'more and more perpendicular' to $(y-x)$ as x approaches y.

In several contributions strict monotonicity is used to prove uniqueness and global stability of interior equilibria under a wide variety of evolutionary dynamics, cf., e.g., Hofbauer & Sandholm [2009], Hofbauer [2000]. Strict monotonicity applied to the relative fitness function f implies

$$(y-x)\cdot(f(y)-f(x)) < 0 \text{ for all } x, y \in S^n,\ x \neq y.$$

Monotonicity has a weak inequality sign. Hofbauer & Sandholm [2009] use the terminology '(strictly) stable games' and these games concur with the games for which (strict) monotonicity holds.

Strict monotonicity implies the defining condition for an interior *ESS*. Strict monotonicity applied to the dynamics, excluding the boundary of the state space, yields the defining condition for an *ESE*. Monotonicity implies that the set of equilibria is connected and convex, strict monotonicity furthermore implies uniqueness of an interior equilibrium, and convergence of various adaptive processes to equilibrium (cf., e.g., Joosten [2006], Harker & Pang [1990]). Among the processes converging to equilibrium under (strict) monotonicity (or local variants thereof) we find BN-dynamics (Nikaidô [1959], Hofbauer [2000]), BR-dynamics and logit dynamics (Hopkins [1999], Hofbauer [2000]) and 'Brownian motions' (cf., Hofbauer [2000]).

The following global property, **attractive monotonicity**, can be thought of as a stronger version of strict monotonicity and it induces the attractive versions of both concepts for interior equilibria:

$$\frac{(y-x)\cdot\left(z^2(y)-z^2(x)\right)}{||y-x||\cdot||z^2(y)-z^2(x)||} < \varepsilon \text{ for all } x, y \in D \subseteq S^n,\ x \neq y.$$

To be precise, we take $D \subseteq S^n$ because throughout this paper z^2 was f or h. For f, the relative fitness function, the small change is not necessary. However, for many evolutionary dynamics the vertices of the unit simplex are fixed points, hence the above is immediately violated if D is allowed to be the entire unit simplex, and the concept would become void. Evidently, z^2 can only have at most one zero on D. Again the interpretation is obvious, the angle between vectors $(y-x)$ and $\left(z^2(y)-z^2(x)\right)$ is never acute and bounded away from 90 degrees. Attractive monotonicity guarantees that if $z^2(y) = 0^{n+1}$, then for all $x, y \in D \subseteq S^n$, $x \neq y$:

$$\frac{(y-x)\cdot\left(z^2(y)-z^2(x)\right)}{||y-x||\cdot||z^2(y)-z^2(x)||} = \frac{-\,(y-x)\cdot z^2(x)}{||y-x||\cdot||z^2(x)||} < \varepsilon.$$

Alternatively, strong monotonicity, cf., e.g., Harker & Pang [1990], implies

$$\frac{(y-x)\cdot\big(z^2(y)-z^2(x)\big)}{||y-x||^2} < \varepsilon \text{ for all } x, y \in int\ S^n,\ x \neq y.$$

A localized variant of this property may yield further results in the spirit of the results obtained here, an elegant interpretation however, seems lacking.

Logit dynamics, componentwise given as follows,

$$h_i^{L,\beta}(x) = e^{\beta f_i(x)} - \left(\sum_{j=1}^{n+1} e^{\beta f_i(x)}\right) x_i \text{ for all } x \in int\ S^n,$$

have a limit for $\beta \to \infty$ in the BR-dynamics. Hofbauer [2000] presents a generalized notion of the BN-dynamics componentwise defined as follows

$$h_i^{BN,\alpha}(x) = (\max\{0, f_i(x)\})^\alpha - \left(\sum_{j=1}^{n+1} (\max\{0, f_j(x)\})^\alpha\right) x_i \text{ for all } x \in int\ S^n.$$

For $\alpha \to \infty$, the dynamics $h^{BN,\alpha}$ are also equal to the BR-dynamics.

There is a point why we mention these dynamics. Figure 2 gives a visualization of dynamics in evolutionary game theory. We showed that there is a one-parameter family of dynamics connecting the replicator and the orthogonal-projection dynamics and another one connecting the latter to the ray-projection dynamics. This insight helped us to obtain several results in the previous section.

Possible extensions of dynamic stability results of the *ESS* are also to be expected for *BN*, *L*, *WL* and *BR*. Thus far only isolated results and proofs exist showing that *ESS* is sufficient for dynamic stability (cf., e.g., Nikaidô [1959], Hofbauer [2000], Hopkins [1999]). By the above it is possible to connect these four types of dynamics with one (or two) parameter families as well joining at *BR*. The aim of future research could then be to extend known results to these families of dynamics. The restriction most probably enabling these results is attractiveness.

7 Conclusion

We presented attractiveness, a refinement criterion to be applied to equilibria in evolutionary game theory. Attractiveness stipulates an upper bound for the angle between a pair of (vector) functions defining the equilibrium concept at hand, in the latter's vicinity. To be more precise, this angle should be acute and strictly bounded away from 90 degrees.

We have applied the criterion to several equilibrium concepts in evolutionary game theory, the (generalized) evolutionarily stable state (Maynard

Smith & Price [1973], Joosten [1996]), the evolutionarily stable equilibrium (Joosten [1996]), the truly evolutionarily stable state and the generalized evolutionarily stable equilibrium (Joosten [2009]).

The viability of any refinement concept hinges on two aspects: the survival rate of equilibria and promised value added of the properties of the refined notion not necessarily shared by the original. On the first one, the equilibria not surviving the refinement criterion are precisely those that are weakly attractive but not attractive. This raises the evasive issue of genericity, but only a very small fraction of evolutionary equilibria seem involved.

The second aspect has been addressed in this paper with interesting first results. Since attractive evolutionary equilibria form a subset of the corresponding concepts, obviously all results on dynamic stability pertaining to the latter must hold for the former as well. Additionally, we showed that certain mis-specifications of the dynamics or the underlying payoff structure are 'harmless', in the sense that attractiveness renders some robustness to results and conclusions about the behavior of the system nearby.

Furthermore, different attractive equilibrium concepts coincide for certain classes of evolutionary dynamics. We showed that the attractive versions of the *ESE* and *ESS* concur for (a subclass of the) barycentric projection dynamics. Also, equivalence was shown of attractive *ESS* and *TESS* for certain families of perturbations of the replicator dynamics. Finally, we demonstrated a similar equivalence between attractive *ESS* and *GESE* with respect to a suitable metric under 'adaptive' dynamics, a large subclass of weakly compatible evolutionary dynamics.

This adds another layer of robustness to results as neither the complete specification of the dynamics and payoff structure nor the equilibrium concept to be used matter for the validity of conclusions about the dynamic system nearby. We wish to emphasize that evolutionary models suffer from several sources of ambiguity, usually assumed away. The dynamics might be known only incompletely, or the payoff structure driving them, or the way payoffs translate into fitness or alternatively, utilities, and the latter into micro-adjustments of agents.

Even if all the aspects mentioned are known completely indeed, it still remains a fact that the dynamics on the aggregate or macro level are deterministic approximations of very complex underlying stochastic processes (cf., e.g., Sandholm [2010b]). Our results indicate that the refinement criterion of attractiveness offers the kind of resilience to cope with all kinds of ambiguities inevitable in the framework of evolutionary game theory.

The present contribution implicitly proposes a framework to look for 'meta' equilibria, i.e., those that satisfy several equilibrium conditions of different nature for the dynamics at hand, where the latter or the payoff structure underlying them can be regarded defined 'roughly'.

8 Appendix

Derivation of barycentric ray-projection dynamics: Let $x \in int\ S^n$ and let $x + \Delta t f(x) \in int\ \mathbb{R}^{n+1}$, then the projection $\widetilde{x}$ of the latter unto the unit simplex S^n along the ray towards $a = \alpha \cdot 1^{n+1}$ for $\alpha \leq 0$ is given by

$$\widetilde{x} = x + \Delta t f(x) - \frac{\Delta t \sum_i^{n+1} f_i(x)}{\sum_i^{n+1} a_i - 1 - \Delta t \sum_i^{n+1} f_i(x)} \left(a - x - \Delta t f(x)\right).$$

Here, Δt is the (sufficiently small) length of the time interval elapsed. Then, with regard to the projection unto the unit simplex this implies a move from $x \in S^n$ to $\widetilde{x} \in S^n$ and therefore

$$\begin{aligned} & \Delta x = \widetilde{x} - x \\ = \ & x + \Delta t f(x) - \frac{\Delta t \sum_{i=1}^{n+1} f_i(x)}{\sum_{i=1}^{n+1} a_i - 1 - \Delta t \sum_{i=1}^{n+1} f_i(x)} \left(a - x - \Delta t f(x)\right) - x \\ = \ & \Delta t \left[f(x) - \frac{\Delta t \sum_{i=1}^{n+1} f_i(x)}{\sum_{i=1}^{n+1} a_i - 1 - \Delta t \sum_{i=1}^{n+1} f_i(x)} \left(a - x - \Delta t f(x)\right) \right]. \end{aligned}$$

As $\dot{x} = \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\Delta t}$, we have

$$\begin{aligned} \dot{x} &= \lim_{\Delta t \downarrow 0} \frac{\Delta t}{\Delta t} \left[f(x) - \frac{\Delta t \sum_{i=1}^{n+1} f_i(x)}{\sum_{i=1}^{n+1} a_i - 1 - \Delta t \sum_{i=1}^{n+1} f_i(x)} \left(a - x - \Delta t f(x)\right) \right] \\ &= \lim_{\Delta t \downarrow 0} \left[f(x) - \frac{\sum_{i=1}^{n+1} f_i(x)}{\sum_{i=1}^{n+1} a_i - 1 - \Delta t \sum_{i=1}^{n+1} f_i(x)} \left(a - x - \Delta t f(x)\right) \right] \\ &= f(x) - \frac{\sum_{i=1}^{n+1} f_i(x)}{\sum_{i=1}^{n+1} a_i - 1} (a - x) = f(x) + \frac{\sum_{i=1}^{n+1} f_i(x)}{1 - \alpha(n+1)} \left(\alpha \cdot 1^{n+1} - x\right). \end{aligned}$$

8.1 Proofs

Lemma 1 We start with the latter part of the statement, i.e., for weak sign-compatibility. For interior states is easy to see that $\sum_{i=1}^{n+1} f_i(x) > 0$ implies that all subgroups with negative relative fitness decrease, hence only subgroups with positive relative fitness can grow, and outside of equilibrium at least one of them must grow. Furthermore, if $\sum_{i=1}^{n+1} f_i(x) \leq 0$, then $h_j^\alpha(x) \geq f_j(x)$ for all $j \in I^{n+1}$. Hence, positive relative fitness implies growth. So, barycentric projection dynamics are weakly sign-compatible. To prove weak compatibility we compute for $x \in int\ S^n$

$$\begin{aligned} f(x) \cdot h^\alpha(x) &= f(x) \cdot \left[f(x) + \frac{\sum_{i=1}^{n+1} f_i(x)}{1 - (n+1)\alpha} (\alpha \cdot 1^{n+1} - x) \right] \\ &= ||f(x)||^2 + \frac{\alpha}{1 - (n+1)\alpha} \left[\sum_{i=1}^{n+1} f_i(x) \right]^2 - x \cdot f(x) \end{aligned}$$

$$
\begin{aligned}
&= ||f(x)||^2 + \frac{\alpha}{1-(n+1)\alpha}\left[\sum_{i=1}^{n+1} f_i(x)\right]^2 \\
&\geq ||f(x)||^2 + \frac{\alpha}{1-(n+1)\alpha}\left(\sqrt{n+1}\cdot ||f(x)||\right)^2 \\
&= ||f(x)||^2 \cdot \left(\frac{1}{1-(n+1)\alpha}\right) \geq 0.
\end{aligned}
$$

Jensen's inequality justifies the first inequality sign above. ■

Proposition 2 Let the interior state y satisfy attractiveness for $z = (z^1, z^2)$ and let U be the open set containing y as stipulated in the definition of attractiveness. Then, we use the trigonometric identity

$$\cos(\beta+\gamma) = \cos\beta\cdot\cos\gamma - \sin\beta\cdot\sin\gamma,$$

as follows. Define $\beta = \sup_{x\in U\setminus\{y\}} \beta(x)$ and $\gamma = \sup_{x\in U\setminus\{y\}} \gamma(x)$ where $\beta(x)$ is the angle between $z^1(x)$ and $z^2(x)$, and $\gamma(x)$ is the angle between $z^2(x)$ and $z(x) \in Z^\theta(z^2, \varepsilon)$, $\theta \in (0,1)$. Then, $\cos\ \beta > \varepsilon$ and $\cos\gamma \geq \sqrt{1-(\theta\varepsilon)^2}$. This in turn implies that

$$
\begin{aligned}
\sup_{x\in U\setminus\{y\}} \frac{z^1(x)\cdot z(x)}{||z^1(x)||\cdot||z(x)||} &= \cos(\beta+\gamma) > \varepsilon\sqrt{1-(\theta\varepsilon)^2} - \theta\varepsilon\sqrt{1-\varepsilon^2} \\
&= \varepsilon\left(\sqrt{1-(\theta\varepsilon)^2} - \sqrt{\theta^2-(\theta\varepsilon)^2}\right) > 0.
\end{aligned}
$$

So, y is also attractive for (z^1, z). ■

Proposition 3 Note that

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} = \frac{||h^{OPD}(x)||}{||f(x)||}\frac{(y-x)\cdot h^{OPD}(x)}{||y-x||\cdot||h^{OPD}(x)||}.$$

Because the *OPD* induce a vector in the plane of the unit simplex and $f(x)$ is always perpendicular to x, we can easily see that

$$\frac{1}{2}\sqrt{2} \leq \frac{||h^{OPD}(x)||}{||f(x)||} \leq 1.$$

Clearly, if y is an attractive *ESE* for the *OPD* (with ε_{ESE} referring to the inequality (A) for *ESE*), then

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \frac{||h^{OPD}(x)||}{||f(x)||}\varepsilon_{ESE} \geq \frac{1}{2}\sqrt{2}\varepsilon_{ESE}.$$

Conversely, if y is an attractive *ESS* (with ε_{ESS} referring to the inequality (A) for *ESS*), then

$$\frac{(y-x)\cdot h^{OPD}(x)}{||y-x||\cdot||h^{OPD}(x)||} > \frac{||f(x)||}{||h^{OPD}(x)||}\varepsilon_{ESS} \geq \varepsilon_{ESS}.$$

This proves the statement of the proposition. ■

Lemma 4 Note that

$$\begin{aligned} & \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \\ = & \frac{(y-x)\cdot f(x)-(y-x)\cdot C(x)\cdot(x-a)+(y-x)\cdot C(x)\cdot(x-a)}{||y-x||\cdot||f(x)||} \\ = & \frac{(y-x)\cdot f(x)-(y-x)\cdot C(x)\cdot(x-a)}{||y-x||\cdot||f(x)||}+C(x)\frac{(y-x)\cdot(x-a)}{||y-x||\cdot||f(x)||} \end{aligned}$$

With $a=\alpha\cdot 1^{n+1}$ and $C(x)=\frac{\sum_{i=1}^{n+1}f_i(x)}{1-(n+1)\alpha}$, the latter equation equals

$$\begin{aligned} & \frac{||h^\alpha(x)||}{||f(x)||}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||}+\frac{C(x)}{||f(x)||}\frac{(y-x)\cdot x}{||y-x||} \\ = & \frac{||h^\alpha(x)||}{||f(x)||}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||}+\frac{\sum_{i=1}^{n+1}f_i(x)}{||f(x)||}\frac{(y-x)\cdot x}{||y-x||\cdot||x||}\frac{||x||}{1-(n+1)\alpha}. \end{aligned}$$

Observe furthermore that $\frac{\sum_{i=1}^{n+1}f_i(x)}{||f(x)||}\leq\frac{||f(x)||_1}{||f(x)||}\leq\frac{\sqrt{n+1}||f(x)||}{||f(x)||}=\sqrt{n+1}$ and $-\sqrt{\frac{1}{2}}\leq\frac{(y-x)\cdot x}{||y-x||\cdot||x||}\leq\sqrt{\frac{1}{2}}$. The former inequalities are standard, the latter one follows immediately from the insight that $y,x\in S^n$. The cosine of the angle between $(y-x)$ and x is therefore in between the values mentioned. Finally, $\sqrt{\frac{1}{n+1}}\leq||x||\leq 1$. So,

$$\frac{||h^\alpha(x)||}{||f(x)||}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||}-\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\in\sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha}(-1,1)\,.$$

This proves the statement of the lemma. ■

Proposition 5 Let y be an attractive *ESE* and let for all $x\neq y$ sufficiently near y :

$$\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||}>\varepsilon_{ESE},$$

then by Lemma 4

$$\begin{aligned} \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} & \geq \frac{||h^\alpha(x)||}{||f(x)||}\varepsilon_{ESE}-\sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha} \\ & \geq \sqrt{\frac{1}{2}}\varepsilon_{ESE}-\sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\alpha}. \end{aligned}$$

Hence, if $\varepsilon_{ESE}-\sqrt{n+1}\frac{1}{1-(n+1)\alpha}>0$, then y must be an *ESS* as well. Then, an upper bound for α to guarantee for the latter to hold is determined by

$$\begin{aligned} \varepsilon_{ESE} &> \sqrt{n+1}\frac{1}{1-(n+1)\alpha} \Longleftrightarrow \frac{\varepsilon_{ESE}}{\sqrt{n+1}} > \frac{1}{1-(n+1)\alpha} \Longleftrightarrow \\ \frac{\sqrt{n+1}}{\varepsilon_{ESE}} &< 1-(n+1)\alpha \Longleftrightarrow \frac{\sqrt{n+1}}{\varepsilon_{ESE}} - 1 < -(n+1)\alpha \\ &\Longleftrightarrow -\left[\frac{\sqrt{n+1}}{(n+1)\varepsilon_{ESE}} - \frac{1}{n+1}\right] > \alpha \Longleftrightarrow \frac{1}{n+1} - \frac{1}{\sqrt{n+1}\varepsilon_{ESE}} > \alpha. \end{aligned}$$

We immediately see the importance of ε_{ESE} here, the less the angle between $(y-x)$ and $h^{\alpha}(x)$ is bounded away from zero, the more this upper bound for α is decreased. Conversely, the more $h^{\alpha}(x)$ points into the direction of y, the less negative α may be. Now, take $\alpha_0 = -\frac{1}{\sqrt{n+1}\varepsilon_{ESE}}$, then the above demonstrates that

$$\begin{aligned} \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} &> \sqrt{\frac{1}{2}}\varepsilon_2 - \sqrt{\frac{n+1}{2}}\frac{1}{1-(n+1)\left(-\frac{1}{\sqrt{n+1}\varepsilon_2}\right)} \\ &= \frac{1}{2}\sqrt{2}\frac{\varepsilon_2^2}{\varepsilon_2+\sqrt{n+1}} > 0. \end{aligned}$$

So, for $\alpha_0 = -\frac{1}{\sqrt{n+1}\varepsilon_{ESE}}$ the state y is both an *ESS* and an *ESE*. We happened to start the proof with taking y as an attractive *ESE*, starting the other way around, i.e., assuming that y is an attractive *ESS*, will yield an upper bound $\alpha_0' = -\sqrt{\frac{n+1}{2\varepsilon_{ESS}^2}}$ expressed in terms of ε_{ESS}. ■

Proposition 6 Observe that for all interior states x, y

$$(y-x)\cdot f(x) = \sum_{i=1}^{n+1}(y_i-x_i)\cdot f_i(x) = \sum_{i=1}^{n+1}\frac{y_i-x_i}{x_i}x_i f_i(x) = \sum_{i=1}^{n+1}\frac{y_i-x_i}{x_i}h_i^{REP}(x).$$

So, if y is an interior *ESS*, it is a *TESS* for the replicator dynamics and vice versa. Let $z_i(x) = \frac{y_i-x_i}{x_i}$, then

$$\begin{aligned} \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} &= \frac{\sum_{i=1}^{n+1}\frac{y_i-x_i}{x_i}\cdot x_i f_i(x)}{||y-x||\cdot||f(x)||} \\ &= \frac{\sum_{i=1}^{n+1} z_i(x)\cdot x_i f_i(x)}{||y-x||\cdot||f(x)||} = \frac{\sum_{i=1}^{n+1} z_i(x)\cdot h_i^{REP}(x)}{||y-x||\cdot||f(x)||} \\ &= \frac{||z(x)||\cdot||h^{REP}(x)||}{||y-x||\cdot||f(x)||}\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||}. \end{aligned}$$

Clearly, for every $U \subseteq int\ S^n$ numbers $m_U = \min_{x\in U}\min_i x_i$ and $m_U = \max_{x\in U}\max_i x_i$ satisfying $0 < m_U \leq M_U$ exist such that for all $x \in U$

$$m_U||y-x|| \leq ||z(x)|| = \sqrt{\sum_{i=1}^{n+1}\left(\frac{y_i-x_i)}{x_i}\right)^2} \leq M_U||y-x||. \qquad \text{(H)}$$

Furthermore, we obtain similarly

$$m_U||f(x)|| \leq ||h^{REP}(x)|| = \sqrt{\sum_{i=1}^{n+1} x_i^2 \cdot f_i^2(x)} \leq M_U||f(x)||.$$

Therefore,

$$\frac{m_U}{M_U}\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||} \leq \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \leq \frac{M_U}{m_U}\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||}$$

Hence, if y is an interior attractive *ESS*, we have that an open neighborhood U exists containing y such that

$$\varepsilon_{ESS} \leq \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \leq \frac{M_U}{m_U}\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||}.$$

So,

$$\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||} \geq \frac{m_U}{M_U}\varepsilon_{ESS}.$$

Define the latter as ε_{TESS}, then we have established a lower bound for the cosine of the angle between $z(x)$ and $h^{REP}(x)$ in $U\backslash\{y\}$.

To prove the converse implication note that

$$\frac{m_U}{M_U}\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \leq \frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||} \leq \frac{M_U}{m_U}\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}.$$

The proof of the converse implication is similar. If y is an interior attractive *TESS* for the replicator dynamics, then

$$\varepsilon_{TESS} < \frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{REP}(x)||} \leq \frac{M_U}{m_U}\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}.$$

So,

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \frac{m_U}{M_U}\varepsilon_{TESS}.$$

Define the latter as ε_{ESS} in this case. This completes the proof. ■

Proposition 7 Let $z(x) = \left(\frac{y_1-x_1}{x_1}, ..., \frac{y_{n+1}-x_{n+1}}{x_{n+1}}\right)$, then

$$\begin{aligned} & z(x)\cdot h^{q-REP}(x) \\ = & \sum_{i=1}^{n+1} \frac{y_i-x_i}{x_i} x_i^q \left(f_i(x) - \frac{\sum_{j=1}^{n+1} x_j^q f_j(x)}{\sum_{j=1}^{n+1} x_j^q}\right) \\ = & \sum_{i=1}^{n+1} (y_i-x_i)\, x_i^{q-1} \left(f_i(x) - \frac{\sum_{j=1}^{n+1} x_j^q f_j(x)}{\sum_{j=1}^{n+1} x_j^q}\right) \end{aligned}$$

$$= (y-x)\cdot h^{(q-1)-REP}(x) + \sum_{i=1}^{n+1}(y_i - x_i)\,x_i^{q-1}\left(\frac{\sum_{j=1}^{n+1} x_j^{q-1} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q-1}} - \frac{\sum_{j=1}^{n+1} x_j^{q} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q}}\right)$$

$$= (y-x)\cdot h^{(q-1)-REP}(x) + (y-x)\cdot x^{q-1}\left(\frac{\sum_{j=1}^{n+1} x_j^{q-1} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q-1}} - \frac{\sum_{j=1}^{n+1} x_j^{q} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q}}\right).$$

Here $x^{q-1} = \left(x_1^{q-1}, ..., x_{n+1}^{q-1}\right)$. Hence,

$$\frac{(y-x)\cdot h^{(q-1)-REP}(x)}{||y-x||\cdot||h^{(q-1)-REP}(x)||} = \frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{q-REP}(x)||}\frac{||h^{q-REP}(x)||}{||h^{(q-1)-REP}(x)||}$$
$$-\frac{(y-x)\cdot x^{q-1}}{||y-x||\cdot||h^{(q-1)-REP}(x)||}\left(\frac{\sum_{j=1}^{n+1} x_j^{q-1} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q-1}} - \frac{\sum_{j=1}^{n+1} x_j^{q} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q}}\right).$$

Taking $q \to 1$, we have

$$\frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{q-REP}(x)||}\frac{||h^{q-REP}(x)||}{||h^{(q-1)-REP}(x)||} -$$
$$\frac{(y-x)\cdot x^{q-1}}{||y-x||\cdot||h^{(q-1)-REP}(x)||}\left(\frac{\sum_{j=1}^{n+1} x_j^{q-1} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q-1}} - \frac{\sum_{j=1}^{n+1} x_j^{q} f_j(x)}{\sum_{j=1}^{n+1} x_j^{q}}\right) \to$$
$$\frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{q-REP}(x)||}\frac{||h^{q-REP}(x)||}{||h^{(q-1)-REP}(x)||} - \frac{(y-x)\cdot x^{0}}{||y-x||\cdot||h^{q-REP}(x)||}\left(\frac{\sum_{j=1}^{n+1} f_j(x)}{n+1}\right) \to$$
$$\frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{(q-1)-REP}(x)||}\frac{||h^{q-REP}(x)||}{||h^{(q-1)-REP}(x)||}.$$

Note that

$$\frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{(q-1)-REP}(x)||}\frac{||h^{q-REP}(x)||}{||h^{(q-1)-REP}(x)||} \to \qquad \text{(CEQ)}$$
$$\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{OPD}(x)||}\frac{||h^{REP}(x)||}{||h^{OPD}(x)||} = \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\frac{||y-x||}{||z(x)||}\frac{||f(x)||}{||h^{OPD}(x)||}.$$

So, if q is sufficiently close to unity, then

$$\frac{z(x)\cdot h^{q-REP}(x)}{||z(x)||\cdot||h^{q-REP}(x)||} > \varepsilon$$

implies that

$$\frac{z(x)\cdot h^{REP}(x)}{||z(x)||\cdot||h^{OPD}(x)||} > \varepsilon(q) > 0 \text{ with } \varepsilon(q) \overset{q\to 1}{\to} \varepsilon.$$

Therefore

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\frac{||y-x||}{||z(x)||}\frac{||f(x)||}{||h^{OPD}(x)||} > \varepsilon(q).$$

Given inequalities (H) in the proof of the previous result and the fact that the orthogonal projection of f and f itself make an angle of at most 45 degrees, we have shown that for $q \to 1$, if y is an attractive *TESS*, the conditions for an attractive *ESS* are fulfilled as well. The other implication can be shown similarly, starting with the final inequality using the central equality (CEQ). ■

Proposition 8 Note that for any interior saturated equilibrium y there exists a neighborhood U'such that for all $x \in U\backslash\{y\}$ and all $i \in I^{n+1}$: $\min\left\{f_i(x)^{2p}, p^{-2p}\right\} = f_i(x)^{2p}$. Hence,

$$h_i^{\alpha,p-PR}(x) = x_i f_i(x) - f_i(x)^{2p} - \frac{\alpha - x_i}{\alpha(n+1)-1}\sum_{j=1}^{n+1} f_j(x)^{2p}.$$

Furthermore, let $f(x)^{2p} = \left(f_1(x)^{2p}, ..., f_{n+1}(x)^{2p}\right)^\top$, then $(y-x)\cdot h^{\alpha,p-PR}(x) = (y-x)\cdot h^{REP}(x) - (y-x)\cdot f(x)^{2p} + (y-x)\cdot x\left(\frac{\sum_{j=1}^{n+1} f_j(x)^{2p}}{\alpha(n+1)-1}\right)$. Consider next

$$\begin{aligned}
&\frac{(y-x)\cdot h^{REP}(x)}{||y-x||\cdot||h^{REP}(x)||}\\
= &\frac{(y-x)\cdot h^{\alpha,p-PR}(x)}{||y-x||\cdot||h^{\alpha,p-PR}(x)||}\frac{||h^{\alpha,p-PR}(x)||}{||h^{REP}(x)||} + \frac{(y-x)\cdot f(x)^{2p}}{||y-x||\cdot||h^{REP}(x)||} - \\
&\frac{(y-x)\cdot x}{||y-x||\cdot||x||}\left(\frac{\sum_{j=1}^{n+1} f_j(x)^{2p}}{\alpha(n+1)-1}\right)\frac{||x||}{||h^{REP}(x)||}\\
= &\frac{(y-x)\cdot h^{\alpha,p-PR}(x)}{||y-x||\cdot||h^{\alpha,p-PR}(x)||}\frac{||h^{\alpha,p-PR}(x)||}{||h^{REP}(x)||} + \frac{(y-x)\cdot f(x)^{2p}}{||y-x||\cdot||f(x)^{2p}||}\frac{||f(x)^{2p}||}{||h^{REP}(x)||} - \\
&\frac{(y-x)\cdot x}{||y-x||\cdot||x||}\left(\frac{\sum_{j=1}^{n+1} f_j(x)^{2p}}{\alpha(n+1)-1}\right)\frac{||x||}{||h^{REP}(x)||}.
\end{aligned}$$

For given $\alpha : \frac{||h^{\alpha,p-PR}(x)||}{||h^{REP}(x)||} \overset{p\to\infty}{\to} 1$; moreover $\frac{||f(x)^{2p}||}{||h^{REP}(x)||} \overset{p\to\infty}{\to} 0$,

$$\begin{aligned}
\frac{\sum_{j=1}^{n+1} f_j(x)^{2p}}{\alpha(n+1)-1}\frac{1}{||h^{REP}(x)||} &\leq \frac{n+1}{\alpha(n+1)-1}\frac{f_M(x)^{2p}}{||h^{REP}(x)||}\\
&\leq \frac{n+1\sqrt{n+1}}{\alpha(n+1)-1} f_M(x)^{2p-1} \overset{p\to\infty}{\to} 0.
\end{aligned}$$

This means that

$$\frac{(y-x)\cdot h^{REP}(x)}{||y-x||\cdot||h^{REP}(x)||} \overset{p\to\infty}{\to} \frac{(y-x)\cdot h^{\alpha,p-PR}(x)}{||y-x||\cdot||h^{\alpha,p-PR}(x)||}.$$

Hence, if the left hand side is larger than $\varepsilon > 0$, the right hand side must be larger than some positive lower bound, too, and vice versa. ■

Proposition 9 Let y be an attractive *GESE* w.r.t. $V_y = (y-x)A^{-1}(y)(y-x)$. So, an open neighborhood of y containing it, and an $\varepsilon_G > 0$ exist such that

$$\frac{(y-x)\cdot A^{-1}(y)h^A(x)}{||y-x||\cdot||A^{-1}(y)h^A(x)||} = \frac{(y-x)\cdot A^{-1}(y)A(x)f(x)}{||y-x||\cdot||A^{-1}(y)A(x)f(x)||} > \varepsilon_G \text{ for all } x \in U\backslash\{y\}.$$

Continuity of A implies that $A(x) \overset{x\to y}{\to} A(y)$, hence an open neighborhood $U' \subset U$ containing y exists such that

$$\frac{1}{2}\varepsilon_G < \frac{(y-x)\cdot A^{-1}(y)A(y)f(x)}{||y-x||\cdot||A^{-1}(y)A(y)f(x)||} = \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}.$$

So, y is an attractive *ESS.* The other implication can be proven similarly.

9 References

Björnerstedt J & J Weibull, 1996, Nash equilibrium and evolution by imitation, in: Arrow KJ et al (eds) "The Rational Foundations of Economic Behavior", MacMillan, London, *pp.* 155-171.

Brown GW & J von Neumann, 1950, Solutions of games by differential equations, *Ann Math Studies* **24**, Princeton University Press, Princeton, *pp.* 73-79.

Friedman D, 1991, Evolutionary games in economics, *Econometrica* **59**, 637-666.

Fudenberg D & DK Levine, 1998, "The Theory of Learning in Games", MIT Press, Cambridge, MA.

Gilboa I & A Matsui, 1991, Social stability and equilibrium, *Econometrica* **59**, 859-868.

Harker PT & JS Pang, 1990, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math Programming 48*, 161-220.

Harper M, 2010, Escort evolutionary game theory, arXiv:0911.1764.

Hofbauer J, 2000, From Nash and Brown to Maynard Smith: Equilibria, dynamics, and ESS, *Selection* **1**, 81-88.

Hofbauer J & WH Sandholm, 2009, Stable games and their dynamics, *J Econ Theory* **144**, 1665-1693.

Hofbauer J, P Schuster & K Sigmund, 1979, A note on evolutionary stable strategies and game dynamics, *J Theoret Biol* **81**, 609-612.

Hofbauer J & K Sigmund, 1998, "Evolutionary Games and Population Dynamics", Cambridge University Press, Cambridge.

Hopkins E, 1999, A note on best response dynamics, *Games Econ Behav*

29, 138-150.
Joosten R, 1996, Deterministic evolutionary dynamics: a unifying approach, *J Evol Econ* **6**, 313-324.
Joosten R, 2006, Walras and Darwin: an odd couple? *J Evol Econ* **16**, 561-573.
Joosten R, 2009, Paul Samuelson's critique and equilibrium concepts in evolutionary game theory, *Papers on Econ & Evol* **#0916**, ISSN 1430-4716.
Joosten R & B Roorda, 2008, Generalized projection dynamics in evolutionary game theory, *Papers on Econ & Evol* **#0811**, ISSN 1430-4716.
Joosten R & B Roorda, 2011, On evolutionary ray-projection dynamics, *Math Meth OR* **74**, 147-161.
Lahkar R & WH Sandholm, 2008, The projection dynamic and the geometry of population games, *Games Econ Behav* **64**, 565-590.
Matsui A, 1992, Best-response dynamics and socially stable strategies, *J Econ Theory* **57**, 343-362.
Maynard Smith J & GA Price, 1973, The logic of animal conflict, *Nature* **246**, 15-18.
Nachbar JH, 1990, 'Evolutionary' selection dynamics in games: Convergence and limit properties, *Intern J Game Theory* **19**, 59-89.
Nikaidô H, 1959, Stability of equilibrium by the Brown-von Neumann differential equation, *Econometrica* **27**, 654-671.
Nikaidô H & H Uzawa, 1960, Stability and non-negativity in a Walrasian tâtonnement process, *Intern Econ Rev* **1**, 50-59.
Perko L, 1991, "Differential Equations and Dynamical Systems", Springer, Berlin.
Ritzberger K & J Weibull, 1995, Evolutionary selection in normal form games, *Econometrica* **63**, 1371-1399.
Sandholm WH, 2005, Excess payoff dynamics and other well-behaved evolutionary dynamics, *J Econ Theory* **124**, 149-170.
Sandholm WH, 2010a, Pairwise comparison dynamics and evolutionary foundations for Nash equilibrium, *Games* **1**, 3-17.
Sandholm WH, 2010b, Population Games and Evolutionary Dynamics, MIT Press, Cambridge, MA.
Schlag K, 1998, Why imitate, and if so: How? A boundedly rational approach to multi-armed bandits, *J Econ Theory* **78**, 138-156.
Schlag K, 1999, Which one should I imitate?, *J Math Econ* **31**, 493-522.
Sethi R, 1998, Strategy-specific barriers to learning and non-monotonic selection dynamics, *Games Econ Behav* **23**, 284-304.
Swinkels J, 1993, Adjustment dynamics and rational play in games, *Games Econ Behav* **5**, 455-484.
Taylor PD & LB Jonker, 1978, Evolutionarily stable strategies and game dynamics, *Math Biosci* **40**, 245-156.
Zeeman EC, 1981, Dynamics of the evolution of animal conflicts, *J Theoret Biol* **89**, 249-270.