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House Allocation with Overlapping Agents: A Dynamic Mechanism Design Approach *

Morimitsu Kurino †

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Abstract

Many real-life applications of house allocation problems are dynamic. For example, in the case of on-campus housing for college students, each year freshmen apply to move in and graduating seniors leave. Each student stays on campus for a few years only. A student is a "newcomer" in the beginning and then becomes an "existing tenant." Motivated by this observation, we introduce a model of house allocation with overlapping agents. In terms of dynamic mechanism design, we examine two representative static mechanisms of *serial dictatorship* (SD) and *top trading cycles* (TTC), both of which are based on an ordering of agents and give an agent with higher order an opportunity to obtain a better house. We show that for SD mechanisms, the ordering that favors existing tenants is better than the one that favors newcomers in terms of Pareto efficiency. Meanwhile, this result holds for TTC mechanisms under time-invariant preferences in terms of Pareto efficiency and strategy-proofness. We provide another simple dynamic mechanism that is strategy-proof and Pareto efficient.

Keywords: house allocation, overlapping agents, dynamic mechanism, top trading cycles, serial dictatorship

JEL classification: C71; C78; D71; D78

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1 Introduction

The static allocation problem[1] of assigning indivisible goods, called "houses," to agents without monetary transfers has been extensively studied and applied to real-life markets such as on-campus housing for college students (cf. Abdulkadiroğlu Sönmez, 1999; Chen and Sönmez, 2002; Guillen and Kesten, 2008), kidney exchanges for patients (Roth, Sönmez, and Ünver 2004), and school choice for U.S. public schools (cf. Abdulkadiroğlu and Sönmez, 2003). Until now, there has been little attempt to analyze dynamic house allocations problems.[2]

Considering dynamic aspects enables us to explain aspects of the allocation problem that cannot be captured by static models. For example, in the case of on-campus housing for college students, each year freshmen apply to move in and graduating seniors leave. Each student stays on campus for a few years only. A student is a "newcomer" in the beginning and then becomes an "existing tenant." In general, students are *overlapping*. In this structure, it is not always *dynamically* Pareto efficient to have a static Pareto efficient allocation in each period.

To illustrate this point, suppose in the first period $t = 1$, there is one agent a^0, called an *initial existing tenant*,[3] who came before the market starts and lives only in this period. Moreover, in each period $t \geq 1$, one agent a^t comes to live in a house in periods t and $t+1$. In each period t, there is an *existing tenant* a^{t-1} who came in the previous period, and a *newcomer* a^t. There are two durable houses h_1 and h_2 available. Each agent prefers h_1 to h_2, and (h_2, h_1) to (h_1, h_2).[4] Consider the allocation:

	$t=1$	$t=2$	$t=3$	$t=4$	$\cdots$
a^0	h_2				
a^1	h_1	h_2			
a^2		h_1	h_2		
a^3			h_1	h_2	
$\vdots$				$\vdots$	$\vdots$

[1]See Sönmez and Ünver (2008) for a recent survey.

[2]See recent exceptions: Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), and Ünver (2009).

[3]Throughout this paper, the terminology "existing tenants" indicates the agents who came to the market in the previous period. It does not always mean that they have property rights for houses, unlike the ones used by Abdulkadiroğlu and Sönmez(1999).

[4]For example, (h_2, h_1) is a consumption path where an agent consumes house h_2 in the first period, and h_1 in the next. Note that this preference violates the discounted utility model. However, considering a critique of the discount utility model as reviewed by Frederick, Lowenstein, and O'Donoghue (2002), we allow for any strict preference relation on $\{h_1, h_2\} \times \{h_1, h_2\}$ in this paper. See footnote 15 for a further discussion.

In each period, an existing tenant is assigned h_2 and a newcomer is assigned h_1. This allocation is Pareto efficient for each period's *static* market. However, consider an infinite exchange between an exiting tenant and a newcomer in each period where an existing tenant exchanges her house h_2 for the newcomer's house h_1. As a result, the initial existing tenant is assigned h_1, and each of the other agents is assigned (h_2, h_1). This new allocation is preferred to the original by every agent. Thus, the original allocation is not dynamically Pareto efficient.

Many universities in the United States use a variant of the *random serial dictatorship mechanism* to allocate dormitory rooms.[5] This mechanism randomly orders the agents and then applies the serial dictatorship (SD) mechanism: the first agent is assigned her top choice, and the next agent is assigned her top choice among the remaining rooms, and so on. This ordering is not entirely random, but rather depends on *seniority*. That is, existing tenants are favored over newcomers.

In the previous example, consider *period orderings* which order a newcomer a^t as the first and an existing tenant a^{t-1} as the second in each period. Running an SD mechanism in each period, we obtain the same allocation as indicated in the previous table. As we saw, this outcome is not dynamically Pareto efficient. On the other hand, consider other period orderings that order an existing tenant first, and a newcomer next. The allocation by the SD mechanism with orderings that favor existing tenants over newcomers Pareto dominates the original, and is dynamically Pareto efficient. That is, the ordering based on seniority performs well in terms of Pareto efficiency.

The subject of this paper is to present a new dynamic framework for a house allocation problem by considering overlapping agents,[6] and to analyze the impact of orderings on Pareto efficiency and strategy-proofness.[7] To our knowledge, the existing literature takes orderings as given, and we are the first to examine the importance of seniority-based mechanisms.

Our model extends the standard overlapping generations (OLG) models[8] to a house allocation problem. Time is discrete and lasts forever. There are a finite number of durable houses that are collectively owned by some institution, say a housing office. In each period, a finite number of newcomers arrive and then stay for a finite number of periods, T, while the oldest agents leave the market. Each agent consumes one house in each period. Each agent has a

[5]We will list some of real-life examples later in this section.

[6]Block and Cantala (2008) independently consider a similar model to ours. One of the difference is that in their model only *one* agent arrives in each period. See the Related literature section for a further discussion.

[7]By strategy-proofness, all agents find it best to report true preferences.

[8]See Samuelson (1958), or Ljungqvist and Sargent (2004).

time-separable preference over houses that spans T periods, consisting of T period preferences. Her given preference does not vary across time. That is, her type is drawn as a preference when she arrives, but her type does not change over time. However, we do allow period preferences to vary across periods. Only initial existing tenants who arrived before the market starts may have endowments. In this environment, a housing office needs to find a mechanism to allocate houses to agents. Unlike in a static model, the office is not able to elicit the preferences of agents who will arrive in the future. In each period, the office assigns houses to agents present in the market, and may assign property rights for the future as well as the current assignment. Thus, the office takes into account the previous assignments in order to determine a current assignment. Hence, the office faces a *dynamic mechanism design* problem.

We study two dynamic mechanisms. The first is a *spot mechanism* where in each period a housing office asks agents present in that period about the current period preference, and not the preference over all time periods. In particular, we look at spot mechanisms with or without property rights transfer: In a spot mechanism *with* property rights transfer, the houses occupied by the oldest agent become vacant in the next period, but those occupied by the other agents become their endowment. On the other hand, a spot mechanism *without* property rights transfer has no such transfer. Another dynamic mechanism is a *futures mechanism* in which each new agent is asked to reveal her preference over all time periods when she arrives.

At any point of time, our spot mechanism without property rights transfer resembles a *house allocation problem* (Hylland and Zeckhauser, 1979). A random serial dictatorship (RSD) (static) mechanism has been studied (Abdulkadiroğlu and Sönmez, 1998). Some colleges such as Davidson College, Lafayyett College, and St. Olaf College use a seniority-based RSD spot mechanism on the condition that all students are forced to participate in the mechanism every year. As we saw in the previous example, in an SD spot mechanism (period orderings are given each period), period orderings that favor existing tenants induce a Pareto efficient allocation (Theorem 2). On the other hand, period orderings that favor newcomers do not always induce a Pareto efficient allocation (Theorem 3).

Although it is simple, Pareto efficient, and strategy-proof (Svensson, 1994), the RSD mechanism is rarely used. Rather, many universities use a modified version of this mechanism, called *a RSD mechanism with squatting rights*, where existing tenants either keep their current rooms, or give up them and participate in the RSD mechanism. The main reason[9] is that universi-

[9] James Earle, Assistant Vice Chancellor for Business at the University of Pittsburgh, gave me the following official reason: The goal of the Department of Housing is first and foremost, customer satisfaction. By allowing students the opportunity to retain a room they like, we are guaranteeing the satisfaction of these returning

ties want to keep students on campus, which makes the universities financially less risky. This seniority-based mechanism is used in Northwestern University, University of Michigan, and the University of Pittsburgh, among others. Students in these colleges can choose stay off-campus. Even colleges that require all students to live on campus use this seniority-based mechanism; for example, Godrdon College, Guilford College, Lawrence University. Although it is ex ante individually rational, this mechanism is not Pareto efficient (Abdulkadiroğlu and Sönmez, 1999), and not ex post individually rational (Some students who participate in the RSD mechanism may get a worse room than their previously owned one.).

The RSD mechanism with squatting rights motivates us to introduce a *spot mechanism with property rights transfer* where, in each period, the houses occupied by the oldest agents become vacant in the next period, but those occupied by the other agents are inherited as property rights or endowments in the next period. At any one point of time, this spot mechanism resembles a *static house allocation problem with existing tenants* (Abdulkadiroğlu and Sönmez, 1999) in which there are newcomers (agents with no endowments) and existing tenants (agents with endowments). In a static context, since the SD with squatting rights is not Pareto efficient, Abdulkadiroğlu and Sönmez (1999) propose a mechanism based on the *top trading cycles (TTC) mechanism* (Shapley and Scarf, 1974), referred to as *AS-TTC mechanism*. This mechanism restores Pareto efficiency that the RSD mechanism with squatting rights lacks, while satisfying individual rationality and strategy-proofness.

We introduce a notion of *acceptability* in which each agent is made weakly better off as time goes on. This corresponds to the situation, in a spot mechanism with property rights transfer, where the static mechanism in each period is individually rational. Thus, it can be seen as a counterpart of individual rationality in a static problem. As we mentioned, in order to keep students on campus, many universities give property rights to students. In this sense, the acceptability is desirable for dynamic mechanisms. However, we prove an Impossibility Theorem where there is no dynamic mechanism that is Pareto efficient and acceptable (Theorem 1).

Any SD spot mechanism is not acceptable, since all houses that an existing tenant weakly prefers to their previously occupied one can be obtained by agents with higher order. However, since an AS-TTC static mechanism is individually rational, we consider a *TTC spot mechanism* in which an AS-TTC static mechanism is run each period in a spot mechanism with property

customers. Furthermore, if these students were forced out of their room, they could not only become a dissatisfied customer, if they then get a room they don't like, but they could also decide to live off campus and become someone else's customer. Why risk the loss of revenue, when you have the potential to have a satisfied customer simply by allowing them to retain their room?

rights transfer. Since an AS-TTC mechanism is individually rational, a TTC spot mechanism is acceptable. However, by the Impossibility Theorem, this spot mechanism is not Pareto efficient.[10] We restrict the preference domain to time-invariant preferences where each agent has preference consisting of identical period preferences. We emphasize that this is not just a repetition of an AS-TTC static mechanism but has two distinct features. First, we have entry and exit of agents with different preferences in each period. Second, endowments or property rights are endogenous. Under time-invariant preferences, we show that period orderings that favor existing tenants perform better than the ones that favor newcomers in terms of Pareto efficiency and strategy-proofness (Theorems 5, 6, 7 and 8).

Finally, we propose a *serial dictatorship (SD) futures mechanism* which is based on orderings of agents. We show that it is strategy-proof and Pareto efficient.

1.1 Related literature

There is an extensive literature on static house allocation problems. See Sönmez and Ünver (2008) for a recent and comprehensive survey.

A dynamic house allocation problem can be classified depending on how and when agents arrive and exit. But with the deterministic arrival and exit of agents, Bloch and Cantala (2008) independently consider a model similar to ours. There are several differences that distinguish our work from theirs. First, in their model only *one* agent enters and exits the market in each period, while our model allows for any finite number of agents to enter and exit. Second, in their model the type of an entering agent is drawn as a period preference but does not vary as time goes on, while in our model we allow period preferences to vary across periods. Third, their preference domain is more restricted than ours. They consider two cases: 1) all agents have identical preferences, and 2) agents have heterogenous preferences but the surpluses from matchings are supermodular. They analyze a Markovian assignment mechanism with property rights transfer that is acceptable. Their seniority-rule corresponds to our *constant SD spot mechanism favoring existing tenants* (to be defined in section 4) in a specific environment as described above. However, they do not look at how static mechanisms such as a serial dictatorship mechanism or a TTC mechanism behave in a dynamic setting. They characterize the independent convergent rule when agents are homogenous, but do not consider incentive issues.

[10] Note that in the general preference domain a TTC spot mechanism is not dynamically but statically Pareto efficient. An SD mechanism with squatting rights is not even statically Pareto efficient.

Ünver(2009) studies a dynamic mechanism design with an application to kidney exchange for patients (Roth, Sönmez, and Ünver, 2004) in which agents arrive stochastically. However, our dynamic model cannot be applied to kidney exchange for two reasons. First, a patient with live donors (i.e. an agent with an endowment) arrives in each period, while in our model only initial existing tenants may have endowments. Second, kidney patients immediately leave the market once their exchange is done, but our model does not allow for this.

Additionally, Abdulkadiroğlu and Loertscher (2007) consider a dynamic problem without the arrival and exit of agents and with two periods in which each agent's type is drawn in each period. They also introduce a *dynamic mechanism* that depends on the first period allocation, and examine efficiency and optimal dynamic mechanisms. Similarly, the case of multiple-type (static) housing markets, where multiple types of indivisible goods are traded and endowments are given, can be seen as a dynamic house allocation problem with finite horizon the length of which is the same as the number of types. Konishi, Quint and Wako (2001) obtain a negative result in which there is no mechanism that is Pareto efficient, individually rational, and strategy-proof.

Although there are almost no papers on how important the ordering of agents is in a mechanism, Sönmez and Ünver (2005) show that a stochastic AS-TTC mechanism favoring newcomers is equivalent to the *core-based mechanism* in a *static* house allocation with exiting tenants.

Finally, there is a growing literature on dynamic mechanism design with monetary transfers. For example, see Athey and Segal (2007) and Gershkov and Moldovanu (2009).

2 The Model

2.1 A dynamic problem

Time is discrete, starts at $t = 1$ and lasts forever. There is a finite set, $\hat{H}$, of indivisible goods, called **houses**, which are collectively owned by some institution (say a housing office.). The houses are perfectly durable in that they can be used in each period. The number of available houses is fixed throughout time.

Agents live in houses for T periods, where $T \geq 2$ is finite.[11] An agent who came before the

[11]If $T = 1$, then our model has a different static model in each period, so there is no dynamic issue. Thus, we exclude $T = 1$.

Table 1: Demographic structures

	1	2	$\cdots$	$T-1$	T	$\cdots$	t	$t+1$	$\cdots$	$t+T-2$	$t+T-1$
a_i^{2-T}	✓										
a_i^{3-T}	✓	✓									
$\vdots$	$\vdots$	$\vdots$	$\ddots$								
a_i^0	✓	✓	$\cdots$	✓							
a_i^1	✓	✓	$\cdots$	✓	✓						
$\vdots$						$\ddots$					
a_i^{t-T+1}							✓				
a_i^{t-T+2}							✓	✓			
$\vdots$							$\vdots$	$\vdots$	$\ddots$		
a_i^{t-1}							✓	✓	$\cdots$	✓	
a_i^t							✓	✓	$\cdots$	✓	✓
$\vdots$											

Note: The mark "✓" indicates when a newcomer a_i^τ in period τ is in the market with the corresponding period in the first row.

model starts is called an **initial existing tenant**.[12] In particular, an initial existing tenant who came at period $\tau \leq 0$ is called a **newcomer in period** τ, and lives in one house in each period from period 1 to $\tau + T - 1$. For example, an oldest agent in period 1 is a newcomer in period $2 - T$, and lives in a house only in period 1. In each period $t \geq 1$, newcomers arrive to live in a house in every period from period t to $t+T-1$. Each such agent is called a **newcomer in period** t. The number of newcomers in each period $t \geq 2 - T$ is finite, and is denoted by n. Let $N(t) := \{a_1^t, a_2^t, \cdots, a_n^t\}$ be the set of newcomers in period $t \geq 2 - T$.[13] Table 2.1 shows the demographic structure of our model. Note that there are both an infinite number of periods and an infinite number of agents in this model. This "double infinity" is the major source of the theoretical peculiarities of the OLG model (Shell, 1971).

In each period $t \geq 1$, agents in the market are newcomers in periods $t - T + 1, t + T + 1, \cdots, t - 1, t$. Agents who came before period t are also called **existing tenants in period** t. That is, they are newcomers in periods $t - T + 1, \cdots, t - 1$. Let $E(t)$ be the set of all existing tenants in period t. Thus, $E(t) \equiv \cup\{N(\tau) : t - T + 1 \leq \tau \leq t - 1\}$ and $E(1)$ is the set of all

[12] See footnote 3.

[13] A variable indexed by (t) is defined only in period t.

initial existing tenants. Moreover, let $A(t) := N(t) \cup E(t)$ be the set of all agents present in period $t \geq 1$. Note $|A(t)| = nT$. We assume that the number of houses is equal to the number of agents present in each period; that is, $|\hat{H}| = |A(t)| = nT$. Throughout this paper, we fix the sets $\hat{H}$ and $A(t)$ for each $t \geq 1$.

For notational simplicity, we introduce a **virtual house**, h_0, which can be assigned to any number of agents. Later we will need to keep track of property rights or endowments assigned in each period. The virtual house will be used to assign no endowment to agents. Let $H := \hat{H} \cup \{h_0\}$. To distinguish houses in $\hat{H}$ from the virtual house, we call a house in $\hat{H}$ a **real house**.

A **period** t **matching**, $\mu(t)$, is an assignment of houses to agents in $A(t)$ such that each agent is assigned one (real or virtual) house and only the virtual house h_0 can be assigned to more than one agent in period $t \geq 1$. For each a in $A(t)$, we refer to $\mu_a(t)$ as the **period** t **assignment** of agent a under $\mu(t)$. Let $\mathcal{M}(t)$ be the set of all period t matchings. A **matching plan** is a collection of period t matchings from period 1 on, denoted by $\mu := \{\mu(t)\}_{t=1}^{\infty}$. For each a in $A(t)$, we refer to $\mu_a := (\mu_a(t), \mu_a(t+1), \cdots, \mu_a(t+T-1))$ as the **assignment** of agent a under μ. Let $\mathcal{M}$ be the set of all matching plans.

Each initial existing tenant, a, in $N(t)$ has a strict preference relation, R_a, on the product H^{T+t-1}. In other words, R_a is a linear order over H^{T+t-1}.[14] Given assignments μ_a and $\hat{\mu}_a$, $\mu_a \, R_a \, \hat{\mu}_a$ means that agent a weakly prefers μ_a to $\hat{\mu}_a$, and $\mu_a \, P_a \, \hat{\mu}_a$ means that agent a strictly prefers μ_a to $\hat{\mu}_a$ under R_a. On the other hand, a newcomer in period $t \geq 1$ has a strict preference relation, R_a, on the product H^T. In addition, we assume that each agent has a time-separable preference defined as follows:[15]

Definition 1. A preference, R_a, of newcomer a in period $t \geq 1$ is **time-separable** if for each $\tau = t, \cdots, T+t-1$, there exist strict preferences $R_a(\tau)$ on H such that for any two assignments μ_a^1 and μ_a^2 on H^T,

$$\text{if } \forall \tau = t, \cdots, t+T-1, \mu_a^1(\tau) \, R_a(\tau) \, \mu_a^2(\tau) \text{ and } \exists \hat{\tau}, \mu_a^1(\hat{\tau}) \, P_a(\hat{\tau}) \, \mu_a^2(\hat{\tau}), \text{ then } \mu_a^1 \, P_a \, \mu_a^2.$$

The above definition is similarly defined for initial existing tenants. Moreover, $R_a(\tau)$ is called

[14] A linear order is a complete, reflexive, transitive, and antisymmetric binary relation.

[15] As we discussed in the Introduction, our assumption of time-separable strict preference violates the discounted utility (DU) model in two ways. Even if her preference is time-invariant, an agent may prefer improving path of houses over declining paths, which violates the DU model. If not so, a period preference in some period may be affected by houses experienced in prior or future periods, which violates the independence assumption of the DU model. We do not go into details of experimental results on the validity of these assumptions. See Frederick, Lowenstein, and O'Donoghue (2002), especially section 4.2.4 and 4.2.5, for further discussions.

a **period** τ **preference**. A preference is called **time-invariant** if all period preferences are identical.

The time-separability condition means that preferences between houses in the same period do not depend on the assignment of houses in the other periods. Moreover, we assume that the virtual house is the worst choice for any period preference of each agent.

We write $\mu_a \, R_a \, \mu'_a$ as $\mu \, R_a \, \mu'$ when no confusion arises. Let $\mathcal{R}_a$ be the set of all preference relations of agent a, and $\mathcal{R} := \prod\{\mathcal{R}_a : a \in A\}$ be the set of all preference profiles. Let $\mathcal{R}_a(\tau)$ be the set of all period τ preferences of agent a, and $\mathcal{R}(\tau) := \prod\{\mathcal{R}_a(\tau) : a \in A(\tau)\}$ be the set of all period preference profiles for agents present in period τ.

An **endowment profile** is expressed by a matching plan $e := \{e_a\}_{a\in A} \in \mathcal{M}$. An endowment of each agent except the initial existing tenants consists only of the virtual house. We consider two cases: 1) each initial existing tenant has an endowment consisting only of the virtual house, 2) each initial existing tenant has the right to live in *one* real house only in period 1; that is, $e_a = (h, h_0, \cdots, h_0)$ for some real house h and for the virtual house h_0. A **house allocation problem with overlapping agents** or simply, a **dynamic problem** is expressed by (A, H, R, e). The first case (second case) in the above is called a **dynamic problem without endowments (with endowments)**. The problem with endowments is considered for a specific mechanism.[16] Unless stated explicitly, a dynamic problem is either with or without endowments. Throughout this paper, we fix A, H.

As we discussed in the Introduction, instead of the pure RSD mechanism, many universities introduce squatting rights in that existing tenants have the right to extend their lease in order to make on-campus housing more attractive. This motivates the following:

Definition 2. In a dynamic problem without endowments, a matching plan $\{\mu(t)\}_{t=1}^{\infty}$ is **acceptable** if each agent is better off as time goes on. That is:

1. $\forall t$ if $2 - T \leq t \leq 0$, $\forall a \in N(t)$, $\forall \tau = t+1, \cdots, t+T-1$, $\mu(\tau)R_a(\tau)\mu(\tau-1)$, and

2. $\forall t \geq 1$, $\forall a \in N(t)$, $\forall \tau = t+1, \cdots, t+T-1$, $\mu(\tau)R_a(\tau)\mu(\tau-1)$.

For a dynamic problem with endowments, it is **acceptable** if the above conditions hold and each initial existing tenant is assigned a house that is at least as good as her endowment in period 1, i.e., $\forall a \in E(1), \mu(1) \, R_a(1) \, e(1)$.

[16]The specific mechanism is a spot mechanism with property rights transfer.

The first condition is for initial existing tenants, and the second is for the other agents.

A matching plan is **Pareto efficient (PE)** if there is no other matching plan that makes all agents weakly better off and at least one agent strictly better off.

3 Dynamic Mechanisms

At any given time, the housing office is not able to ask newcomers who will arrive in the future about their preference. In order to reflect preferences, the office cannot determine the houses from the beginning to the future at once. Instead, it determines the assignment in each period. This feature brings about new aspects for mechanism design problems. First, message spaces can take many forms even if we focus on *direct* mechanisms. For example, in each period, the office can ask an agent about her corresponding period preference or her entire preference. Second, in each period, the office can assign not only the houses for the current period but also houses for the future. Finally, in any given period, some of the houses are already assigned, and thus the office has to take into account this past assignment in order to decide on the current assignment.

Generally, for a dynamic problem with or without endowments, a **dynamic mechanism** is a function $\Pi : \mathcal{R} \to \mathcal{M}$ that determines a matching plan for each preference profile. A dynamic mechanism is **acceptable** if it always selects an acceptable matching plan. Moreover, it is **Pareto efficient** if it always selects a Pareto efficient matching plan.

We restrict attention to two dynamic mechanisms. The first is a **spot mechanism** where, in each period, the office asks each agent present in the period to reveal her corresponding *period* preference. We also consider a **futures mechanism**. In the first period the office asks all agents present in this period (i.e. initial existing tenants and newcomers in period 1) about their preference. In subsequent periods, the office asks newcomers about their entire preference.

3.1 Spot mechanisms

We formally define a spot mechanism by introducing the concepts of a *static problem* and a *static mechanism.*

3.1.1 Static mechanisms

Fix a dynamic problem with or without endowments, (A, H, R, e). Consider a period $t \geq 1$. A **period t static problem** is defined as $(D(t), U(t), H, R(t), e(t))$.[17] An agent a in $D(t)$ is called an **endowed agent** and occupies a real house, $e_a(t)$, while an agent in $U(t)$ is called an **unendowed agent** and does not have the right to live in any real house. Thus, agents in $A(t)$ are either endowed or unendowed, that is, $A(t) \equiv D(t) \cup U(t)$. A real house is called **vacant** if it is not occupied by any endowed agent. Moreover, $R(t)$ is a strict preference profile of agents in $A(t)$, and $e(t)$ is the endowment profile of agents in $A(t)$.

In such models, if $D(t) = \emptyset$ and $U(t) = A(t)$, a static problem is a house allocation problem (Hylland and Zeckhauser, 1979). If $D(t) = A(t)$ and $U(t) = \emptyset$, it is a housing market (Shapley and Scarf, 1974). Finally, if $D(t) \neq \emptyset$, $U(t) \neq \emptyset$, and $D(t) \cup U(t) = A(t)$, it is a house allocation problem with existing tenants (Abdulkadiroğle and Sömez, 1999).

In a static problem where there are endowed agents, a matching is **individually rational** if no endowed agent strictly prefers her endowment to her assignment. A matching is **Pareto efficient** if there is no other matching that makes all agents weakly better off and at least one agent strictly better off.

A **period t static mechanism** determines a period t matching for each of both a period preference profile and an endowment profile. That is, it is a function $\gamma^t : \mathcal{R}(t) \times \mathcal{M}(t) \to \mathcal{M}(t)$. It is denoted by $\gamma^t(R(t), e(t))$ for each period preference profile, $R(t)$, and each endowment profile, $e(t)$. A period t static mechanism is **individually rational** (**Pareto efficient**) if it always selects an individually rational (Pareto efficient) period t matching. In addition, it is **strategy-proof** if truth-telling is a weakly dominant strategy in its associated preference revelation game.

3.1.2 Spot mechanisms without property rights transfer

In this section, we define a spot mechanism without property rights transfer. To make the mechanism consistent with the problem, we look only at a dynamic problem without endowments. In this mechanism, $D(t) = 0$ and $U(t) = A(t)$ for each $t \geq 1$, we always have a static house allocation problem in each period. This motivates the following definition:

Definition 3. Given a sequence of static mechanisms $\{\gamma^t\}_{t=1}^{\infty}$, a **spot mechanism without**

[17] Recall that a variable indexed by (t) is defined only in period t.

property rights transfer, $\Pi : \mathcal{R} \to \mathcal{M}$ with $R \mapsto \Pi(R) := (\Pi(R;1), \Pi(R;2), \cdots) \in \mathcal{M}$, is obtained through static mechanisms as follows: for each period $t \geq 1$,

$$\Pi(R;t) := \gamma^t(R(t), e(t)),$$

where all $e(t)$'s consist of the virtual house.

3.1.3 Spot mechanisms with property rights transfer

Unlike in the previous mechanism, we consider the transfer of property rights in either a dynamic problem with or without endowments. In each period, the houses occupied by the oldest agents become vacant in the next period, but those occupied by the other agents are inherited as property rights or endowments in the next period.

In a dynamic problem with endowments, we have $\forall t \geq 1$, $D(t) = E(t)$. That is, endowed agents are the existing tenants. On the other hand, in a dynamic problem without endowments, we have $D(1) = \emptyset$ and $U(1) = A(1)$, but $\forall t \geq 2$, $D(t) = E(t)$. In other words, in the first period, there is no endowed agent, but endowed agents are the existing tenants from the second period on.

Definition 4. Given a sequence of static mechanisms $\{\gamma^t\}_{t=1}^{\infty}$, a **spot mechanism with property rights transfer**, $\Pi : \mathcal{R} \to \mathcal{M}$ with $R \mapsto \Pi(R) := (\Pi(R;1), \Pi(R;2), \cdots) \in \mathcal{M}$, is obtained through static mechanisms as follows: for each preference profile R in $\mathcal{R}$,

1. In period 1,

$$\Pi(R;1) \equiv \mu(1) := \gamma^1 \left(R(1), e(1)\right).$$

2. In period $t \geq 2$, set $\hat{e}(t) := \left(\mu_{E(t)}(t-1), e_{N(t)}(t)\right)$,

$$\Pi(R;t) \equiv \mu(t) := \gamma^t \left(R(t), \hat{e}(t)\right).$$

A spot mechanism is thus defined by using a sequence of static mechanisms. The link between the period $t-1$ static mechanism and the period t static mechanism is made possible through the endogenous endowment $\hat{e}(t)$ for $t \geq 2$. This makes it different from just a repetition of a static mechanism. In each period, the current period mechanism depends on the previous mechanism through the assigned endowment. More precisely, in period 1, the office faces a period 1 static market whose endowment corresponds to $e(1)$ from the original dynamic problem. The office asks each agent a present in period 1 about her period 1 preference, $R_a(1)$. Based on the reported period preference profile, $R(1)$, and the endowment, $e(1)$, the office determines a period 1 matching $\Pi(R;1) \equiv \mu(1)$ through the period 1 static mechanism, $\gamma^1(R(1), e(1))$. In the next period, $t = 2$, each agent a who is still in the market (i.e., in $E(2)$) has the right to live in the previously assigned house, $\mu_a(1)$. The agents' endowment profile is $\mu_{E(2)}(1)$. Newcomers in period 2 have the virtual endowment. Their endowment profile is $e_{N(2)}(2)$. Thus, we have an endowment profile $\hat{e}(2) := (\mu_{E(2)}(1), e_{N(2)}(2))$ for the period 2 static market. Based on the reported period 2 preference profile, $R(2)$, and the endowment, $\hat{e}(2)$, the office determines a period 2 matching $\Pi(R;2) \equiv \mu(2)$ through the period 2 static mechanism, $\gamma^2(R(2), \hat{e}(2))$. Repeating this process produces the matching plan $\Pi(R) \equiv \{\mu(t)\}_{t=1}^{\infty}$.

3.2 Strategy-proofness

Definition 5. A spot or futures mechanism $\Pi : \mathcal{R} \to \mathcal{M}$ is **strategy-proof** if

$$\forall a \in A,\ \forall R \in \mathcal{R},\ \forall R'_a \in \mathcal{R}_a, \quad \Pi(R_a, R_{-a})\, R_a\, \Pi(R'_a, R_{-a}).$$

Consider a spot mechanism. Agents face an extensive form with simultaneous moves. We are interested in whether they reveal their true period preferences in each period static mechanism. In any given period, revealing the true period preference for an agent does not depend on history, but rather on that period alone. Implicit in the above definition is the restriction of our attention to a class of history-independent strategies. That is, a spot mechanism is strategy-proof if, for each agent, her history-independent strategy of revealing her true period preferences is weakly better than any other history-independent strategy, regardless of the history-independent strategies of the other agents.[18]

On the other hand, a futures mechanism is strategy-proof if truth-telling is a weakly dominant strategy for each agent.

[18]In the definition of strategy-proofness in a *static* mechanism, truth-telling is a weakly dominant strategy for each agent. However, to our knowledge, there is no existing definition of strategy-proofness investigated in our dynamic setting. Instead of requiring truth telling to be a weakly dominant strategy for each agent, we introduce a weaker notion by looking at a class of history-independent strategies in a spot mechanism.

3.3 Impossibility result

In this section, we begin with a negative result. We will investigate positive results in later sections. First, we search for a dynamic mechanism that is Pareto efficient and strongly individually rational. The following result rules out the existence of such a mechanism.

Theorem 1. *Consider a dynamic problem with or without endowments. Suppose there are at least two newcomers in each period who live for at least three periods. Then, there is no dynamic mechanism that is Pareto efficient and acceptable.*

Proof. First, we consider a dynamic problem without endowments. Pick two newcomers a_1^1 and a_2^1 in period 1. Consider the case where agents live for three periods. Their period preferences satisfy (from best to worst):

a_1^1			a_2^1		
$R_a(1)$	$R_a(2)$	$R_a(3)$	$R_a(1)$	$R_a(2)$	$R_a(3)$
h_1	h_2	h_1	h_2	h_2	h_1
h_2	h_1	h_2	h_1	h_1	h_2

For example, $h_1 R_{a_1^1}(1) h_2$. Moreover,

$$a_1^1 : (h, h_2, h_2) \succ (h, h_1, h_1),$$
$$a_2^1 : (h, h_1, h_1) \succ (h, h_2, h_2),$$

where h is any real house. Also, h_1 and h_2 are less preferred to the other houses for the agents other than a_1^1 and a_2^1.

Seeking a contradiction suppose there is a Pareto efficient and acceptable matching plan $\mu \equiv \{\mu(t)\}_{t=1}^{\infty}$. First, by Pareto efficiency, in period 1, a_1^1 is assigned h_1, and a_2^1 is assigned h_2. Then, in period 2, a_2^1 is assigned h_2 from acceptability, and thus a_1^1 is assigned h_1. Similarly, a_1^1 is assigned h^1 from acceptability, and thus a_2^1 is assigned h_2. In summary, we have the following matching plan.

$\mu\vert_{\{a_1^1, a_2^1\}}$	$t=1$	$t=2$	$t=3$
a_1^1	h_1	h_1	h_1
a_2^1	h_2	h_2	h_2

Consider an exchange between a_1^1 and a_2^1 in periods 2 and 3. As a result, a_1^1 is assigned (h_1, h_2, h_2) and a_2^1 is assigned (h_2, h_1, h_1). These assignments are strictly preferred by both agents. Therefore, we have a contradiction.

In the other case where $T > 2$, we can include the previous case to obtain the result. For a dynamic problem with endowments, let a_1^1 own h_1 and a_2^1 own h_2 in the first period. We can do the same argument for this case.

□

4 Serial Dictatorship (SD) Spot Mechanisms

In this section, we consider a spot mechanism without property rights transfer for a dynamic problem without endowments.

4.1 Definition

Since all of the mechanisms examined in this paper are based on an ordering of agents, here we introduce various types of orderings. Given a set $B \subset A$ of agents, an **ordering** in B is a linear order, denoted by f_B. We often denote it as the ordered list:

$$f_B := (b_1, b_2, \cdots, b_m) \text{ if and only if } b_1 \, f_B \, b_2 \, f_B \cdots f_B \, b_m.$$

We say that b_1 is the **first agent in** B, b_2 is the **second agent in** B and so on. In addition, **agent** a **has higher order than agent** b if $a \, f_B \, b$. Specifically, we look at two kinds of orderings. The first is a **period** t **ordering**, $f_{A(t)}$, which is an ordering of $A(t)$, the set of all agents present in period $t \geq 1$. The second is **cohort orderings** $f_{E(1)}$ and $f_{N(t)}$ for $t \geq 1$, where $f_{E(1)}$ is an ordering of $E(1)$ which is the set of all initial existing tenants, while $f_{N(t)}$ is an ordering of $N(t)$ which is the set of all newcomers in period $t \geq 1$.

A **serial dictatorship (SD) spot mechanism** is a spot mechanism without property rights transfer in which each period static mechanism is a **serial dictatorship (SD) static mechanism**. An SD period t static mechanism is based on a period t ordering, $f_{A(t)}$, and is defined as follows. Take any period t ordering, $f_{A(t)}$. Fix a preference profile, $R(t)$. The first agent gets her top choice, the second agent gets her top choice among houses excluding the one assigned to the first agent. The kth agent gets her top choice among houses excluding those assigned to all agents with higher order than her.

It is known that an SD static mechanism is strategy-proof and Pareto efficient (Svensson, 1994). Note that an SD static mechanism is independent of the previously occupied houses. As a result, an existing tenant is not guaranteed to obtain a house that is at least as good as her occupied house in the previous period. Hence, an SD spot mechanism is not acceptable.

4.2 Strategy-proofness

We know that an SD *static* mechanism is strategy-proof and Pareto efficient. The question is whether these properties hold for an SD spot mechanism.

Proposition 1. *In a dynamic problem without endowments, an SD spot mechanism is strategy-proof.*

Proof. An SD spot mechanism has no transfer of property rights and consists of SD static mechanisms, and thus each SD period mechanism is independent of the past assignments. Hence, we have the desired result. □

4.3 Pareto efficiency: some positive results

In a specific example in the Introduction, we demonstrated Pareto efficiency of an allocation induced by an SD spot mechanism favoring existing tenants. In this subsection, we study under what kind of period orderings the induced SD spot mechanism can achieve Pareto efficiency. To this end, we introduce the following:

Definition 6.
1. A **period t ordering $f_{A(t)}$ favors existing tenants** if, in period t, each existing tenant has higher order than any newcomer in $f_{A(t)}$. Moreover, it **favors newcomers** if, in period t, each newcomer has higher order than any existing tenant in $f_{A(t)}$.
2. **A sequence of period orderings favors existing tenants (newcomers)** if, in each period t, a period t ordering favors existing tenants (newcomers).

An SD spot mechanism induced by a sequence of period orderings favoring existing tenants (newcomers) is called a **SD spot mechanism favoring existing tenants (newcomers)**.

Definition 7. A sequence of period orderings is **constant** if the relative ranking of agents is the same across periods. That is, if an agent, a, has higher order than another agent, a', in some period, then a has higher order than a' in any other period when they are in the market.

An SD spot mechanism induced by a constant sequence of period orderings is called a **constant SD spot mechanism**. Now, we can state one of the main positive results.

Theorem 2. *In a dynamic problem without endowments, a constant SD spot mechanism favoring existing tenants is Pareto efficient.*

Before proving the theorem, we explore the relation between the period orderings and the cohort orderings for a given constant sequence of periods orderings favoring existing tenants. The following example illustrates this.

Example. Consider a situation where there are two newcomers, a_1^t and a_2^t, in each period $t \geq -1$. Agents live for three periods. Then, $E(1) = \{a_1^{-1}, a_2^{-1}, a_1^0, a_2^0\}$, $N(t) = \{a_1^t, a_2^t\}$, and $A(t) = N(t-2) \cup N(t-1) \cup N(t)$. Take a sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period orderings such that

$$\begin{aligned} f_{A(1)} &= \left(a_1^0,\, a_1^{-1},\, a_2^0,\, a_2^{-1},\, a_1^1,\, a_2^1\right), \\ f_{A(2)} &= \left(a_1^0,\, a_2^0,\, a_1^1,\, a_2^1,\, a_1^2,\, a_2^2\right), \\ f_{A(3)} &= \left(a_1^1,\, a_2^1,\, a_1^2,\, a_2^2,\, a_1^3,\, a_2^3\right). \end{aligned}$$

Notice that this sequence is constant and favors existing tenants. We can take the following cohort orderings:

$$\begin{aligned} g_{E(1)} &= (a_1^0,\, a_1^{-1},\, a_2^0,\, a_2^{-1}), \\ g_{E(1)}|_{A(2)} &= (a_1^0,\, a_2^0), \\ g_{N(t)} &= (a_1^t,\, a_2^t), \qquad \text{for each } t = 1, 2, 3. \end{aligned}$$

Notice that

$$\begin{aligned} f_{A(1)} &= (g_{E(1)}, g_{N(1)}), \\ f_{A(2)} &= (g_{E(1)}|_{A(2)}, g_{N(1)}, g_{N(2)}), \\ f_{A(3)} &= (g_{N(1)}, g_{N(2)}, g_{N(3)}). \end{aligned}$$

The corresponding cohort orderings are denoted by using f instead of g. □

In summary, we have the following lemma (the proof is straightforward).

Lemma 1. *Given a constant sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period ordering favoring existing tenants, there are corresponding cohort orderings $f_{E(1)}$ and $\{f_{N(t)}\}_{t\geq 1}$ such that*

$$\begin{aligned} f_{A(t)} &= (f_{E(1)}|_{A(t)}, f_{N(1)}, \cdots, f_{N(t)}), \quad \forall t = 1, \cdots, T-1, \text{ and} \\ f_{A(t)} &= (f_{N(t-T+1)}, \cdots, f_{N(t)}), \quad \forall t \geq T. \end{aligned}$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let $\{f_{A(t)}\}_{t=1}^{\infty}$ be given. From Lemma 1, let a sequence $(f_{E(1)}, \{f_{N(t)}\}_{t=1}^{\infty})$ give the corresponding cohort orderings. Let $\mu = \{\mu(t)\}_{t=1}^{\infty}$ be a matching plan generated by a constant SD spot mechanism for some arbitrary preference profile R. To find a contradiction, suppose some matching plan ν Pareto dominates μ. Then,

$$\forall a \in A, \nu R_a \, \mu \quad \text{and } \exists b \in A, \nu P_b \, \mu.$$

Since $A = E(1) \cup (\cup_{t=1}^{\infty} N(t))$, either $b \in E(1)$ or $b \in N(t)$ for some $t \geq 1$. We consider two cases:

Case 1: $b \in E(1)$.

Take an agent $c \in E(1)$ who has the highest order among agents in $\{b \in E(1) : \nu \, P_b \, \mu\}$ with respect to $f_{E(1)}$. Then, since preferences are strict, it follows that

$$\forall a \in E(1) \text{ who has higher order than } c \text{ in } f_{E(1)}, \nu_a = \mu_a. \tag{1}$$

Let $\tau \leq 0$ such that $c \in N(\tau)$. Now it is sufficient to show $\forall t = 1, \cdots, T-1+\tau$, $\mu(t) \, R_c(t) \, \nu(t)$. It then follows from time-separable preferences that $\mu R_c \, \nu$, which is a contradiction. For each period t, for an SD static mechanism, given Lemma 1 and (1), we have that there is no room for agent c to be strictly better off than $\mu_c(t)$. Hence, $\mu(t) \, R_c(t) \, \nu(t)$.

Case 2: $b \notin E(1)$ and $b \in N(t)$, for some $t \geq 1$.

Take the smallest $\tau \geq 1$ such that $\exists b \in N(\tau)$ with $\nu \, P_b \, \mu$. Choose an agent $c \in N(\tau)$ who has the highest order among agents in $\{b \in N(\tau) : \nu \, P_b \, \mu\}$ with respect to $f_{N(\tau)}$. Then, it follows from strict preferences that

$$\begin{aligned} &\forall a \in E(1) \cup \left(\cup_{t=1}^{\tau-1} N(t)\right), \ \nu_a = \mu_a, \text{ and} \\ &\forall a \in N(\tau) \text{ who has higher order than } c \text{ in } f_{N(\tau)}, \ \nu_a = \mu_a. \end{aligned} \tag{2}$$

Now, it is sufficient to show that $\forall t = \tau, \cdots, \tau + T - 1$, $\mu(t) \, R_c(t) \, \nu(t)$. It then follows from time-separable preferences that $\mu R_c \nu$, which is a contradiction. For each period t, for an SD static mechanism, given Lemma 1 and (2), we have that there is no room for agent c to be strictly better off than $\mu_c(t)$. Hence, $\mu(t) \, R_c(t) \, \nu(t)$.

□

4.4 When is an SD spot mechanism undesirable?

As we saw in the Introduction, Pareto efficiency depends on the ordering structure.

Theorem 3. *In a dynamic problem without endowments, an SD spot mechanism favoring newcomers is not Pareto efficient even under time-invariant preferences.*

Here **time-invariant preferences** mean that each agent has a time-invariant preference.

Proof. We extend the demonstration in the Introduction to the case where agents live for any finite period $T \geq 2$. Suppose agents have time-invariant preferences and live for T periods. Fix a sequence of period orderings that favors newcomers. Pick the first agent, a^t, in $f_{A(t)}$ among newcomers in period $t \geq 1$. Each agent a has the same time-invariant preference, where house h_1 is her top choice and house h_2 is her second top choice, such that

$$(h_2, h_1, \mu_a^{t+2}) P_a (h_1, h_2, \mu_a^{t+2}). \tag{3}$$

where μ_a^{t+2} is any assignment of agent a from period $t+2$ to $t+T-1$. Then, an SD mechanism favoring newcomers assigns houses (without parentheses below) to agents a^t, $t \geq 1$, as follows.

	$t=1$	$t=2$	$t=3$	$t=4$	$\cdots$
a^1	h_1	h_2 (h_1)	$\cdots$		
a^2		h_1 (h_2)	h_2 (h_1)	$\cdots$	
a^3			h_1 (h_2)	h_2 (h_1)	$\cdots$
$\vdots$				$\vdots$	$\vdots$

Consider an infinite exchange of houses h_1 and h_2 between the existing tenant a^{t-1} and the newcomer a^t for $t \geq 2$, keeping the assignments of the other agents be the same. This exchange is shown as houses inside the parentheses on the above table. It follows from (3) that the resulting allocation Pareto dominates the induced matching plan. □

5 Top Trading Cycles (TTC) Spot Mechanisms

In this section, we consider a spot mechanism with property rights transfer for a dynamic problem with or without endowments.

5.1 Definition

In a spot mechanism with property rights transfer, we have a house allocation problem with existing tenants in each period. As an example, a random serial dictatorship mechanism with squatting rights is widely used in on-campus housing for college students. In a static setting, a deterministic serial dictatorship mechanism with squatting rights is not Pareto efficient. To restore Pareto efficiency while satisfying individual rationality and strategy-proofness, Abdulkadiroğlu and Sönmez (1999) propose a mechanism referred to as **Abdulkadiroğlu and Sönmez's top trading cycles (AS-TTC) static mechanisms**: Fix a period t ordering, $f_{A(t)}$. For any announced preference profile, $R(t)$, and an endowment profile, $e(t)$, the AS-TTC static mechanism selects a matching through the following **AS-TTC algorithm**:

Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until an agent a demands house $h_{a'}$ of an endowed agent. If at that point the endowed agent whose house is demanded is already assigned a house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting the agent in question to the top and continue the procedure. Similarly, insert any endowed agent who is not already served at the top of the line once her house is demanded. If at any point a loop forms, it is formed by exclusively endowed agents and each of them demands the house of the endowed agent next in the loop. (A **loop** is an ordered list of agents, $(a_1, a_2, \cdots, a_k)$, where agent a_1 demands the house of agent a_2, agent a_2 demands the house of agent a_3, $\cdots$, agent a_k demands the house of a_1.) In such cases, remove all agents in the loop by assigning them the houses they demand and proceed.

Theorem 4 (Abdulkadiroğlu and Sönmez, 1999)**.** *For any ordering, $f_{A(t)}$, the induced AS-TTC static mechanism is Pareto efficient, individually rational, and strategy-proof.*

Note that, from the above procedure, any AS-TTC static mechanism is individually rational. This is because an endowment of an endowed agent will not be assigned to another agent before this endowed agent is assigned a house. If another agent demands the endowment of this endowed agent, she will be promoted to the top of the ordering. While at the top of the ordering, if there is no house available better than her endowment, then existing tenants demand her own house. At this point, a trivial loop consisting of this agent will form, and she will leave and be assigned at worst her own endowment.

A **top trading cycles (TTC) spot mechanism** is a spot mechanism with property rights transfer in which each period static mechanism is an AS-TTC static mechanism, given a sequence of period orderings. Clearly, this TTC spot mechanism is acceptable, since an AS-TTC static mechanism is individually rational in each period.

5.2 Strategy-proofness: some positive results

We know from Theorem 4 that an AS-TTC *static* mechanism satisfies individual rationality, strategy-proofness, and Pareto efficiency. Because acceptability, which can be seen as a counterpart of individual rationality, always holds for a TTC spot mechanism, the question is whether these properties hold in our dynamic problem. By Impossibility Theorem 1, a TTC spot mechanism is not Pareto efficient in general. To answer the above question, we restrict the preference domain to time-invariant preferences. *Throughout this section below, we assume that each agent has a time-invariant preference.*[19] We emphasize that this is not just a repetition of an AS-TTC static mechanism for a corresponding static problem. There are two features that differ from a static problem. First, we have both entry and exit of different agents in each period. Second, the endowment is endogenous. One of our main positive results is the following.

Theorem 5. *Consider a dynamic problem with endowments and time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof among all agents except initial existing tenants. It can be manipulated by initial existing tenants, provided there are at least three newcomers in each period who live for at least three periods.*

Before proving the theorem, we review some of the new concepts stated in Theorem 5. We say that a spot mechanism $\Pi : \mathcal{R} \rightarrow \mathcal{M}$ is **strategy-proof among all agents except initial existing tenants** if for each agent a who is not an initial existing tenant, $\forall R \in \mathcal{R}$, $R'_a \in \mathcal{R}_a$, $\Pi(R_a, R_{-a}) \, R_a \, \Pi(R'_a, R_{-a})$. As in strategy-proofness, we restrict attention to a class of history-independent strategies.

To prove the theorem, we introduce some additional concepts of *effective ordering* introduced by Sönmez and Ünver (2005). For each ordering $f_{A(t)}$, the AS-TTC algorithm assigns houses in one of two possible ways:

1. There is a sub-order $(a^1, \cdots, a^k)$ of agents where a^1 demands the house of a^2, a^2 demands the house of a^3, $\cdots$, agent a^{k-1} demands house of a^k, and a^k demands any available house. We call such a sub-order a **serial-order** (S).
2. There is a sub-order $(a^1, \cdots, a^k)$ of endowed agents where a^1 receives a^k's house, a^k receives a^{k-1}'s house, $\cdots$, a^2 receives a^1's house. Recall that we call such sub-order a **loop-order** (L).

[19]Under this assumption, it is sufficient in a spot mechanism that the housing office asks each agent about her period preference, not in all periods, *once* when she arrives. Also, we can allow the office to do so in each period. We take the former. Even if we use the latter approach, all results are not affected.

For a given ordering, $f_{A(t)}$, construct the **effective ordering**, e_t, as follows: Run the AS-TTC algorithm and order agents in the order their assignments are finalized. When there is a loop-order, order these agents as in the loop-order.

Note that a matching produced by an AS-TTC algorithm with the effective ordering yields the same outcome produced by an SD static mechanism induced by this effective ordering. Also note that the effective ordering is endogenous, depending on preferences and the exogenous ordering $f_{A(t)}$.

We now examine how effective orderings behave under time-invariant preferences for a constant sequence of period orderings favoring existing tenants. Fix a preference profile R and a constant sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period orderings that favors existing tenants. Let $(f_{E(1)}, \{f_{N(t)}\}_{t=1}^{\infty})$ be a sequence of its corresponding cohort orderings. For convenience, we use

$$f_{N(t)} := (a_1^t, a_2^t, \cdots, a_n^t)$$

for each $t \geq 1$. Observe that in period 1,

$$e_1 = \left(\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)}, \overbrace{\underbrace{S}_{a_1^1}, \underbrace{S}_{a_2^1}, \cdots, \underbrace{S}_{a_n^1}}^{N(1)} \right) = \left(\overbrace{X, \cdots, X}^{E(1)}, f_{N(1)} \right),$$

where X is either S or L. Recall that S stands for a serial-order and L for a loop-order. That is, initial existing tenants are before newcomers, since newcomers do not have any endowment and the ordering $f_{A(1)}$ favors existing tenants. Moreover, because each newcomer has no endowment, she will point to an available house and form a serial-order consisting of herself.

Now, consider period 2. First, existing tenants in period 2 (who are in $E(2)$) have higher order than newcomers in the effective ordering e_2. Second, in period 1, initial existing tenants prefer their assignment to those assigned to agents in $N(1)$. Since their assignment becomes an endowment in period 2, it follows from time-invariant preferences that initial existing tenants never point to the houses of agents in $N(1)$ in the algorithm. This implies that initial existing tenants have higher order than agents in $N(1)$ in e_2. Third, since period orderings are constant, agent a_1^1 never points to agent a_i^1 ($i \geq 2$), but points to her own house or an available house. The same applies for other agents in $N(1)$. In summary,

$$e_2 = \left(\overbrace{\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)\cap A(2)}, \overbrace{\underbrace{X}_{a_1^1}, \cdots, \underbrace{X}_{a_n^1}}^{N(1)}}^{E(2)}, \overbrace{\underbrace{S}_{a_1^2}, \cdots, \underbrace{S}_{a_n^2}}^{N(2)} \right) = \left(\overbrace{X, \cdots, X}^{E(1)\cap A(2)}, f_{N(1)}, f_{N(2)} \right).$$

That is, each existing tenant in $N(1)$ forms either a trivial loop-order consisting of herself, or a serial order in which all agents in the serial-order receive a better house than their assignment received in period 1. On the other hand, each newcomer forms a serial-order consisting of herself in the algorithm, because she does not have any endowment.

Repeating this process, in period $\tau = 2, \cdots, T-1$,

$$e_\tau = \left(\overbrace{\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)\cap A(\tau)}, \overbrace{\underbrace{X}_{a_1^1}, \cdots, \underbrace{X}_{a_n^1}}^{N(1)}, \cdots, \overbrace{\underbrace{X}_{a_1^{\tau-1}}, \cdots, \underbrace{X}_{a_n^{\tau-1}}}^{N(\tau-1)}}^{E(\tau)}, \overbrace{\underbrace{S}_{a_1^\tau}, \cdots, \underbrace{S}_{a_n^\tau}}^{N(\tau)} \right),$$

$$= \left(\overbrace{X, \cdots, X}^{E(1)\cap A(\tau)}, f_{N(1)}, \cdots, f_{N(\tau-1)}, f_{N(\tau)} \right).$$

Similarly, in period $\tau \geq T$,

$$e_\tau = \left(\overbrace{\overbrace{\underbrace{X}_{a_1^{\tau-T+1}}, \cdots, \underbrace{X}_{a_n^{\tau-T+1}}}^{N(\tau-T+1)}, \cdots, \overbrace{\underbrace{X}_{a_1^{\tau-1}}, \cdots, \underbrace{X}_{a_n^{\tau-1}}}^{N(\tau-1)}}^{E(\tau)}, \overbrace{\underbrace{S}_{a_1^\tau}, \cdots, \underbrace{S}_{a_n^\tau}}^{N(\tau)} \right),$$

$$= \left(f_{N(\tau-T+1)}, \cdots, f_{N(\tau-1)}, f_{N(\tau)} \right).$$

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Fix a preference profile, R. Consider any agent, a, who is not an initial existing tenant. Consider any other preference, $\hat{R}_a$. Let $\mu := \{\mu(t)\}_{t=1}^\infty$ and $\hat{\mu} := \{\hat{\mu}(t)\}_{t=1}^\infty$ be matching plans induced by a constant TTC spot mechanism favoring existing tenants for (R_a, R_{-a}) and $(\hat{R}_a, R_{-a})$. In each period t from $(\hat{R}_a, R_{-a})$, when agent a is in the market, the effective ordering of agents who have higher order than agent a is not affected. Thus, agent a can get a house that makes her indifferent or worse than $\mu_a(t)$. That is, $\mu_a(t)\, R_a(t)\, \hat{\mu}_a(t)$. By time-separability of preferences, $\mu_a\, R_a\, \hat{\mu}_a$. This completes the proof of the first part.

For the second part, suppose there are at least three newcomers in each period $t \geq 2-T$. They live for at least three periods, T. Fix a constant sequence $\{f_{A(t)}\}_{t=1}^\infty$ of period orderings that favors existing tenants. In light of the proof of the first part, we focus on initial existing tenants. Pick agents a_i^{2-T} and a_i^{3-T}, $i = 1, 2, 3$, such that

$$f_{A(1)}|_{\{a_i^{2-T}, a_i^{3-T} : i=1,2,3\}} := (a_1^{2-T}, a_2^{2-T}, a_3^{2-T}, a_1^{3-T}, a_2^{3-T}, a_3^{3-T}),$$
$$f_{A(2)}|_{\{a_i^{3-T} : i=1,2,3\}} := (a_1^{3-T}, a_2^{3-T}, a_3^{3-T}).$$

Note that a_i^{2-T} lives only in period 1, and a_i^{3-T} lives only in periods 1 and 2, $i = 1, 2, 3$. Period preferences satisfy the table on the left hand side (from best to worst):

a_1^{2-T}	a_2^{2-T}	a_3^{2-T}	a_1^{3-T}	a_2^{3-T}	a_3^{3-T}	a
h_1	h_5	h_3	h_6	h_1	h_1	h
			h_4	h_2	h_2	h_3, h_4, h_5
					h_6	h_1, h_2, h_6

a_2^{3-T}
h_1
h_6

where h is any real house other than houses h_1 to h_6, and a is an agent except a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$. The above table means that agent a prefers h to any of h_3, h_4, h_5, and prefers any of h_3, h_4, h_5 to any of h_1, h_2, h_6. Moreover, the a_2^{3-T}'s preference satisfies

$$(h_6, h_1)\, P_{a_2^{3-T}}\, (h_2, h_2),$$

Table 2: Assignments under the truthful preference (left) and the manipulated preference (right)

	$t = 1$	$t = 2$	$\cdots$
a_1^{2-T} (h_1)	h_1		
a_2^{2-T} (h_2)	h_5		
a_3^{2-T} (h_3)	h_3		
a_1^{3-T} (h_4)	h_4	h_6	
$\boldsymbol{a_2^{3-T}}$ (h_5)	$\boldsymbol{h_2}$	$\boldsymbol{h_2}$	
a_3^{3-T} (h_6)	h_6	h_1	
$\vdots$		$\vdots$	

	$t = 1$	$t = 2$	$\cdots$
a_1^{2-T} (h_1)	h_1		
a_2^{2-T} (h_2)	h_5		
a_3^{2-T} (h_3)	h_3		
a_1^{3-T} (h_4)	h_4	h_6	
$\boldsymbol{a_2^{3-T}}$ (h_5)	$\boldsymbol{h_6}$	$\boldsymbol{h_1}$	
a_3^{3-T} (h_6)	h_2	h_2	
$\vdots$		$\vdots$	

Endowments are indicated with the parentheses in the first column on Table 2.

We will see that a_2^{3-T} manipulates the mechanism by reporting the preference described on the right hand side of the above table. At period 1, whether a_2^{3-T} manipulates or not, any agent a, who is not a_i^{2-T}, a_i^{3-T} $i = 1, 2, 3$, never points to houses h_1 to h_6 in an AS-TTC algorithm. Thus, we concentrate on a restricted static market consisting of agents a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$ and

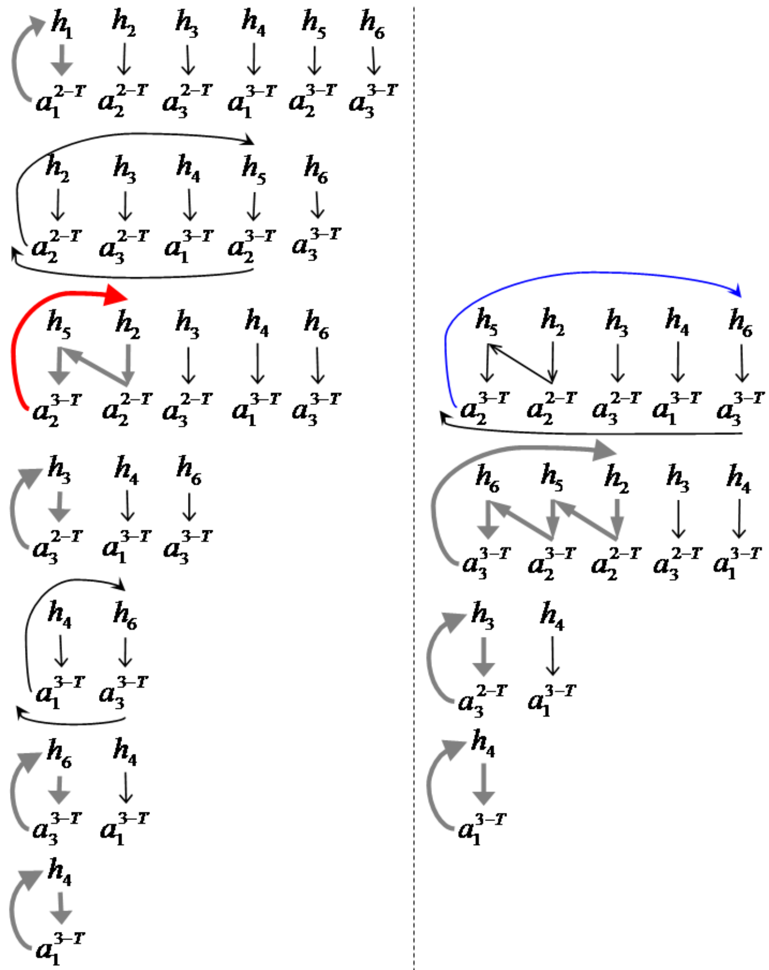

Figure 1: AS-TTC algorithms in period $t = 1$ under the truthful preference (left) and the manipulated preference (right). Thick arrows indicate a cycle in each step.

houses h_1 to h_6 in the algorithm. The procedures to obtain period 1 matchings are illustrated in Figure 1.

In the next period $t = 2$, a_1^{3-T} owns h_4, and a_2^{3-T}, a_3^{3-T} own h_2, h_6. In an AS-TTC algorithm, whether a_2^{3-T} manipulates or not, any agent who are not $a_1^{3-T}, a_2^{3-T}, a_3^{3-T}$ never points to h_1, h_2, h_6. Thus, the first agent among $a_1^{3-T}, a_2^{3-T}, a_3^{3-T}$ who points a house is a_1^{3-T} by either being pointed by the other agent or not. The procedure after a_1^{3-T} has an opportunity to choose a house is depicted in Figure 2.

The resulting assignments for agents a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$ are described in Table 2. Thus, a_2^{3-T} obtains an assignment (h_6, h_1) from lying, while she obtains a worse assignment (h_2, h_2)

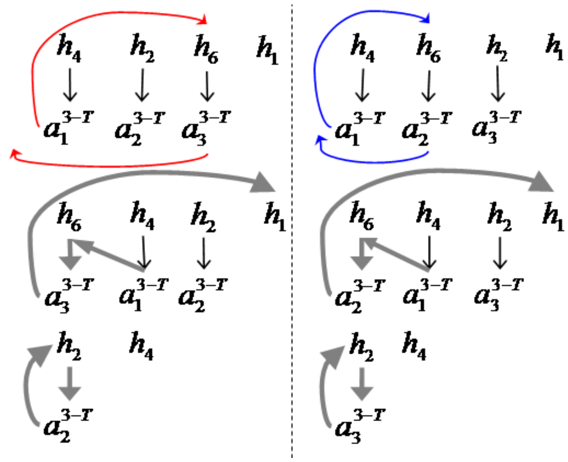

Figure 2: AS-TTC algorithms in period $t = 2$ under the truthful preference (left) and the manipulating preference (right). Thick arrows indicate a cycle in each step.

from truth-telling.

Finally, consider why agent a_2^{3-T} manipulates the mechanism. Given that agent a_1^{3-T} points to h_6 in $t = 2$ and an agent whose assigned house is assigned h_6 in $t = 1$ and becomes an endowment in $t = 2$ can be upgraded in $t = 2$, agent a_2^{3-T} lies so that she can obtain a worse house, h_6, in $t = 1$, but a better house, h_1, at $t = 2$. □

We state two corollaries:

Corollary 1. *Consider a dynamic problem with endowments and time-invariant preferences. Suppose each agent lives for two periods. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof.*

Proof. Initial existing tenants live for only one period. Since the static mechanism is strategy-proof, truth-telling is a dominant strategy for each initial existing tenant. □

Corollary 2. *Consider a dynamic problem without endowments and with time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof.*

Proof. Note that the induced effective ordering takes the same form for agents, except initial existing tenants as the one in a dynamic problem with endowments. Thus, consider an induced effective ordering $e_\tau|_{E(1)\cap A(\tau)}$ restricted to initial existing tenants for each period $\tau = 1, \cdots, T-1$. Similar arguments to the above lead to $e_\tau|_{E(1)\cap A(\tau)} = f_{E(1)}|_{A(\tau)}$. Using the same logic as

Theorem 5, we can conclude that the history-independent strategy of true period preferences is weakly better off than any other history-independent strategy for each initial existing tenant. □

5.3 How can a TTC spot mechanism be manipulated by agents who are not initial existing tenants?

Remember that an SD spot mechanism is strategy-proof. This is because, in each period, it ignores the past assignment. On the other hand, a TTC spot mechanism guarantees each agent a house that is at least weakly better than the previously assigned house. This opens up the possibility of manipulation in which an agent obtains a worse house than she can obtain in truth-telling, expecting her to be upgraded in an ordering by being pointed out by some other agent in the next period. As we saw, a constant TTC spot mechanism favoring existing tenants effectively excludes such a possibility. However, this is not the case if it favors newcomers.

Theorem 6. *Consider a dynamic problem with time-invariant preferences either with endowments or without endowments. Suppose there are at least two newcomers in each period. Then, a TTC spot mechanism favoring newcomers is not strategy-proof among all agents except initial existing tenants.*

Proof. Suppose there are at least two newcomers in each period $t \geq 2-T$. Agents live for T periods. Pick two newcomers a_1^t and a_2^t in each period. Fix a sequence of period orderings that favors newcomers. Without loss of generality, a_1^t has higher order than a_2^t in each period t. Period preferences $R_{a_i^t}(t)$ $(t = 2-T, 3-T, 2, 3;\ i = 1, 2)$ of each agent a_i^t satisfy the table on the left hand side (from best to worst):

a_1^{2-T}	a_2^{2-T}	a_1^{3-T}	a_2^{3-T}	a_1^2	a_2^2	a_1^3	a_2^3
h_2	h_3	h_1	h_4	h_1	h_1	h_2	h_4
				h_3	h_2		
				h_2			

a_1^2
h_1
h_2
h_3

For the other agents, houses h_1 to h_4 are less preferred to any other house. Moreover, agent a_1^2's preference satisfies

$$(h_2, h_1, \mu^4_{a_1^2}) P_{a_1^2} (h_3, h_3, \mu^4_{a_1^2}),$$

where $\mu^4_{a_1^2}$ is any assignment of agent a_1^2 from period 4 on. Unspecified preferences are assumed to be arbitrary.

Endowments are indicated by the parentheses in the first column on the table below. If $T = 2$, agents a_1^{3-T} and a_2^{3-T} are not initial existing tenants but newcomers in period 1. However, the allocations to be calculated will not be affected, even for the case without endowments, because of preferences.

In each period, we concentrate on a static problem consisting of agents $a_i^{2-T}, a_i^{3-T}, a_i^2, a_i^3$, $i = 1, 2$, since houses h_1 to h_4 are less preferred to any other house for each of the other agents.

We will see that agent a_1^2 manipulates the mechanism by reporting the preference described on the right hand side of the above table.

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_2)	h_2			
a_2^{2-T} (h_3)	h_3			
a_1^{3-T} (h_1)	h_1	h_1		
a_2^{3-T} (h_4)	h_4	h_4		
$\vdots$		$\vdots$		
$\boldsymbol{a_1^2}$		$\boldsymbol{h_3}$	$\boldsymbol{h_3}$	$\cdots$
a_2^2		h_2	h_1	$\cdots$
a_1^3			h_2	$\cdots$
a_2^3			h_4	$\cdots$
$\vdots$				$\ddots$

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_2)	h_2			
a_2^{2-T} (h_3)	h_3			
a_1^{3-T} (h_1)	h_1	h_1		
a_2^{3-T} (h_4)	h_4	h_4		
$\vdots$		$\vdots$		
$\boldsymbol{a_1^2}$		$\boldsymbol{h_2}$	$\boldsymbol{h_1}$	$\cdots$
a_2^2		h_3	h_3	$\cdots$
a_1^3			h_2	$\cdots$
a_2^3			h_4	$\cdots$
$\vdots$				$\ddots$

The left hand side shows an allocation by the TTC spot mechanism when agent a_1^2 reveals her true preference, while the right hand side shows an allocation by the TTC spot mechanism when a_1^2 lies. Note that, whether a_1^2 has higher order than a_2^2 in the period 3 ordering or not, the above assignments are not affected. The procedures to obtain each allocation for period 3 static markets are illustrated in Figure 3 in the case that a_1^2 has higher order than a_2^2 in the period 3 ordering.

Thus, a_1^2 obtains an assignment $(h_2, h_1, h_1, \cdots, h_1)$ from lying, while she obtains a worse assignment $(h_3, h_3, \mu^4_{a_1^2})$ from truth-telling, where $\mu^4_{a_1^2}$ is some assignment of a_1^2 from period 4 on.

Consider why agent a_1^2 manipulates the mechanism. Given that newcomer a_1^3 points to h_2 in $t = 3$, and an agent whose assigned house is assigned h_2 in $t = 2$ and becomes an endowment in $t = 3$ can be upgraded in $t = 3$, agent a_1^2 lies so that she can obtain a worse h_2 in $t = 2$, but a better house h_1 in $t = 3$.

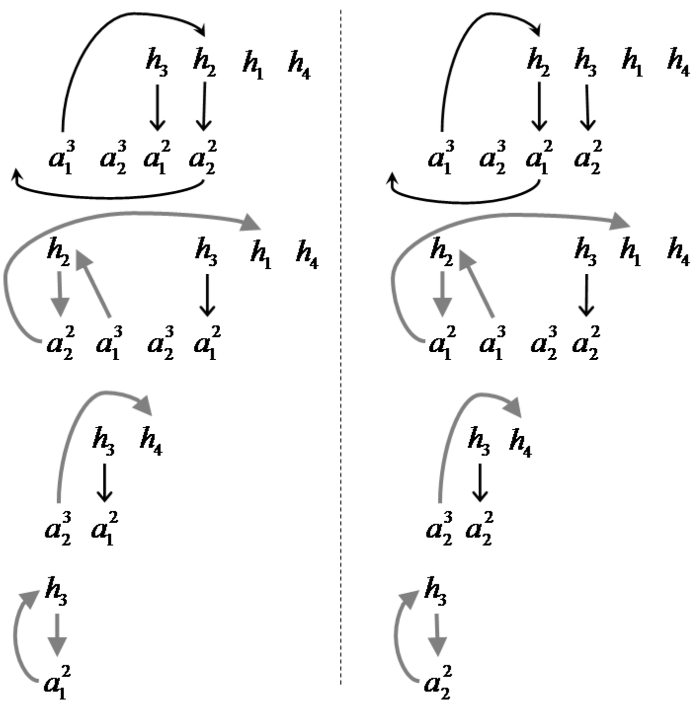

Figure 3: AS-TTC algorithms in period $t = 3$ under the truthful preference (left) and the manipulating preference (right). Thick arrows indicate a cycle in each step.

□

The reason for the failure of strategy-proofness in the previous proof is that, provided that a newcomer has a favorable house, by lying, an existing tenant obtains this house in the previous period and in the next period she gets a better house by being pointed by the newcomer. As we saw in Theorem 5, such an opportunity for all agents except initial existing tenants is excluded by making period orderings favor existing tenants.

To contrast a TTC spot mechanism favoring newcomers with the one favoring existing tenants and an SD spot mechanism, see the last section of the Summary.

5.4 Pareto efficiency: some positive results

We now turn our attention to Pareto efficiency. We saw in the previous subsection that strategy-proofness among all agents except initial existing tenants makes a difference between a constant TTC spot mechanism favoring newcomers and the one favoring existing tenants. Similarly, we introduce a weaker notion for Pareto efficiency.

Definition 8. A matching plan ν **Pareto dominates** another matching plan μ **among all agents except initial existing tenants** if

1. $\{\mu_a(t) : a \in A \setminus E(1)\} = \{\nu_a(t) : a \in A \setminus E(1)\}$ for each $t \geq 1$, and
2. $\forall a \in A \setminus E(1)$, $\nu R_a \mu$ and $\exists a \in A \setminus E(1)$, $\nu P_a \mu$.

Moreover, a matching plan is **Pareto efficient among all agents except initial existing tenants** if it is not Pareto dominated by any other matching plan among all agents except initial existing tenants.

As with strategy-proofness, a constant TTC spot mechanism favoring existing tenants is Pareto efficient among all agents except initial existing tenants, but not Pareto efficient.

Theorem 7. *Consider a dynamic problem with endowments and time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient among all agents except initial existing tenants, but not Pareto efficient, provided there are at least two newcomers in each period who live for at least three periods.*

Proof. For the first part, let $\mu = \{\mu(t)\}_{t=1}^{\infty}$ be a matching plan generated by a constant TTC spot mechanism favoring existing tenants for some arbitrary preference profile, R. Let $\{e_t\}_{t=1}^{\infty}$ be a corresponding sequence of effective orderings. To find a contradiction, suppose some matching plan, ν, Pareto dominates μ among all agents except initial existing tenants. Then,

$$\forall a \in A \setminus E(1), \nu R_a \mu \quad \text{and } \exists a \in A \setminus E(1), \nu P_a \mu.$$

Since $A \setminus E(1) \equiv \cup_{t=1}^{\infty} N(t)$, take the smallest $\tau \geq 1$ such that $\exists a \in N(\tau)$, $\nu P_a \mu$. It follows from strict preferences that

$$\forall t \leq \tau - 1, \forall a \in N(t), \nu_a = \mu_a. \tag{4}$$

In addition, take an agent $b \in N(\tau)$ who has the highest order among agents in $\{a \in N(\tau) : \nu P_a \mu\}$. Then, it follows from strict preferences that

$$\forall a \in N(\tau) \text{ who has a higher order than } b \text{ does}, \nu_a = \mu_a. \tag{5}$$

Now, it is sufficient to show that $\forall t = \tau, \cdots, \tau + T - 1$, $\mu(t) R_b(t) \nu(t)$, since this leads to a contradiction, namely, that $\mu R_b \nu$ and $\nu P_b \mu$. For each $t = \tau, \cdots, \tau + T - 1$, it follows from (4) and (5) that in the effective ordering e_t, each agent, a, ordered before agent b has $\nu_a(t) = \mu_a(t)$. Thus, the AS-TTC algorithm implies that there is no room for agent b to be strictly better off than $\mu_b(t)$. Hence, $\mu(t) R_b(t) \nu(t)$.

For the second part, suppose there are at least two newcomers in each period who live for at least three periods, T. Fix a constant sequence, $\{f_{A(t)}\}_{t=1}^{\infty}$, of period orderings that favors existing tenants. Pick initial existing tenants a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, and a_2^{3-T} such that

$$f_{A(1)}|_{\{a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, a_2^{3-T}\}} = (a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, a_2^{3-T}), \text{ and } f_{A(2)}|_{\{a_1^{3-T}, a_2^{3-T}\}} = (a_1^{3-T}, a_2^{3-T}).$$

Note that a_i^{2-T} lives only in period 1, and a_i^{3-T} lives only in period 1 and 2, $i = 1, 2$. Each agent $a_i^t \neq a_2^{2-T}$ has an identical preference (from best to worst):

$$P_{a_i^t}(t) : h_1, h_2, h_3.$$

Agent a_2^{2-T}'s top choice is h_4. For the other agents, houses h_1 to h_4 are less preferred to any other house. Moreover,

$$(h_2, h_2)\, P_{a_1^{3-T}}\, (h_3, h_1) \quad \text{and} \quad (h_3, h_1)\, P_{a_2^{3-T}}\, (h_2, h_2).$$

Endowments are indicated in the first column on the table below.

The induced TTC spot mechanism produces the following assignments:

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_1)	h_1			
a_2^{2-T} (h_2)	h_4			
a_1^{3-T} (h_3)	h_3 (h_2)	h_1 (h_2)		
a_2^{3-T} (h_4)	h_2 (h_3)	h_2 (h_1)		
$\vdots$				

Consider another matching plan in which a_1^{3-T} exchanges the first two periods assignments (h_3, h_1) for (h_2, h_2) with a_2^{3-T}. This exchange is described by houses inside the parentheses on the above table. This matching plan Pareto dominates the one induced by the TTC spot mechanism.

□

We state two corollaries:

Corollary 3. *Consider a dynamic problem with endowments and time-invariant preferences. Suppose each agent lives for two periods. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient.*

Proof. Pareto efficiency among all agents except initial existing tenants does not consider any matching that involves an exchange between initial existing tenants and the other agents. However, when agents live for two periods, initial existing tenants live for only one period. Since a static AS-TTC spot mechanism is Pareto efficient, any other matching plan involving such a exchange necessarily hurts the initial existing tenants. Note that this logic does not work for the case where agents live for at least three periods. Thus, any matching plan induced by a constant TTC spot mechanism favoring existing tenants is Pareto efficient.

□

Corollary 4. *Consider a dynamic problem without endowments and with time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient.*

Proof. The same argument applies on the induced effective ordering as the one in Corollary 2. □

5.5 When is a TTC spot mechanism undesirable?

In an example taken up in Theorem 3 that shows Pareto inefficiency in an SD spot mechanism favoring newcomers, we demonstrated that an infinite exchange between existing tenants and newcomers Pareto dominates a matching plan induced by the SD spot mechanism. Looking at this example closely, we might think that acceptability precludes such an infinite exchange. Since a TTC spot mechanism satisfies acceptability, one might conjecture that a TTC spot mechanism favoring newcomers is Pareto efficient. However, this is not the case, as shown in the following theorem:

Theorem 8. *Consider a dynamic problem with time-invariant preferences either with endowments or without endowments. Suppose there are at least two newcomers in each period. Then, a TTC spot mechanism favoring newcomers is not Pareto efficient among all agents except initial existing tenants.*

Proof. Suppose there are at least two newcomers in each period $t \geq 2 - T$. They live for T periods. Pick two newcomers a_1^t and a_2^t in each period. Fix a sequence of period ordering

$\{f_{A(t)}\}_{t=1}^{\infty}$ that favors newcomers. Without loss of generality, a_1^t is the first agent in $f_{A(t)}$ in each period. Note that this sequence may not be constant; e.g., a_1^t may not be the first in the subsequent periods. Period preferences satisfy: For each $m \geq 0$,

a_1^{2Tm+2}	a_2^{2Tm+2}	a_1^{2Tm+3}	a_2^{2Tm+3}	$a_1^{T(2m+1)+2}$	$a_2^{T(2m+1)+2}$	$a_1^{T(2m+1)+3}$	$a_2^{T(2m+1)+3}$
h_2	h_2	h_3	h_3	h_1	h_1	h_4	h_4
	h_4		h_2		h_3		h_1
	h_1		h_4		h_2		h_3
	h_3		h_1		h_4		h_2

a_2^{2Tm+2}	a_2^{2Tm+3}	$a_2^{T(2m+1)+2}$	$a_2^{T(2m+1)+3}$
(h_3, μ_a, h_4)	(μ_a, h_1, h_2)	(h_4, μ_a, h_3)	(μ_a, h_2, h_1)
(h_1, μ_a, h_1)	(μ_a, h_4, h_4)	(h_2, μ_a, h_2)	(μ_a, h_3, h_3)

where $\mu_a \in H^{T-2}$ is an arbitrary assignment. Moreover,

a_1^{2-T}	a_2^{2-T}	a_1^{3-T}	a_2^{3-T}
h_1	h_2	h_3	h_4
h	h	h	h

where h is an arbitrary house other than the second row in each column. In the above tables, each column indicates the corresponding preference where an upper house is strictly preferred to the lower one. For any other agent not specified above, houses h_1 to h_4 are less preferred to any other house.

Endowments are indicated by the parentheses in the first column in Table 3. If $T = 2$, agents a_1^{3-T} and a_2^{3-T} are not initial existing tenants but newcomers in period 1. However, the allocations to be calculated will not be affected, even for the case without endowments, because of preferences.

In each period, we concentrate on a static problem consisting of agents a_i^{2-T}, a_i^{3-T}, a_i^{2Tm+2}, a_i^{2Tm+3}, $a_i^{T(2m+1)+2}$, and $a_i^{T(2m+1)+3}$ for $i = 1, 2$ and $m \geq 0$, since houses h_1 to h_4 are less preferred to any other house for each of the other agents.

The induced TTC spot mechanism produces the matching plan μ, where houses without parentheses are the assignments, on Table 3. Note that this matching plan is not affected by whether a sequence of period orderings is constant or not.

Table 3: Matching plans in Theorem 8

	1	2	3	$\cdots$	$T+1$	$T+2$	$T+3$	$\cdots$	$2T+1$	$2T+2$	$2T+3$	$\cdots$
$a_1^{2-T}(h_1)$	h_1											
$a_2^{2-T}(h_2)$	h_2											
$a_1^{3-T}(h_3)$	h_3	h_3										
$a_2^{3-T}(h_4)$	h_4	h_4										
$\vdots$		$\vdots$										
a_1^2		h_2	h_2	$\cdots$	h_2							
a_2^2		h_1	h_1	$\cdots$	$h_1(h_4)$							
a_1^3			h_3	$\cdots$	h_3	h_3						
a_2^3			h_4	$\cdots$	$h_4(h_1)$	$h_4(h_2)$						
$\vdots$						$\vdots$						
a_1^{T+2}						h_1	h_1	$\cdots$	h_1			
a_2^{T+2}						$h_2(h_4)$	h_2	$\cdots$	$h_2(h_3)$			
a_1^{T+3}							h_4	$\cdots$	h_4	h_4		
a_2^{T+3}							h_3	$\cdots$	$h_3(h_2)$	$h_3(h_1)$		
$\vdots$										$\vdots$		
a_1^{2T+2}										h_2	h_2	$\cdots$
a_2^{2T+2}										$h_1(h_3)$	h_1	$\cdots$
a_1^{2T+3}											h_3	$\cdots$
a_2^{2T+3}											h_4	$\cdots$
$\vdots$												$\vdots$

Consider an infinite exchange depicted by houses inside the parentheses in Table 3. Clearly, the resulting allocation Pareto dominates the induced matching plan μ among all agents except initial existing tenants. $\square$

See the last section of the Summary to contrast a TTC spot mechanism favoring newcomers with the one favoring existing tenants and an SD spot mechanism.

6 Serial Dictatorship (SD) Futures Mechanisms

In this section, we consider a dynamic problem without endowments and propose a simple futures mechanism termed **serial dictatorship (SD) futures mechanism**. Fix a sequence $(f_{A(1)}, \{f_{N(t)}\}_{t\geq 2})$ of an ordering of initial agents and an ordering of newcomers in period $t \geq 2$. For any announced preference profile, R, an SD futures mechanism finds a matching plan by using the following algorithm.

Period 1: The first agent in $f_{A(1)}$ gets her top *assignment* (consisting of houses up to the period when she leaves the market) under her reported preference. The kth agent in $f_{A(1)}$ gets her top assignment excluding the houses assigned to all agents before her under her reported preference. This produces the current matching and the future assignment for agents in $A(1)$.

Period t: Given the assignment determined in past periods, each of the existing tenant is assigned a house according to her assignment as previously determined. The first newcomer in $f_{N(t)}$ gets her top assignment excluding the houses assigned to the existing tenants under her reported preference. A kth newcomer in $f_{N(t)}$ gets her top assignment excluding the houses assigned to all existing tenants and all newcomers before this agent under her reported preference. The procedure for newcomers generates the current matching and the future assignment for agents in $A(t)$.

As such, we have the following.

Theorem 9. *In a dynamic problem without endowments, an SD futures mechanism is strategy-proof and Pareto efficient, but not acceptable under the same assumptions as Impossibility Theorem 1.*

Proof. First, we show strategy-proofness. In period 1, the first agent in $f_{A(1)}$ cannot do better by reporting any other preference, since she already receives her top assignment under her reported preference. The kth agent in $f_{A(1)}$ cannot do better than reporting her true preference, since the house distributed until the kth agent is independent of her preference and receives her top assignment among the remaining houses. The argument for any other period is similar.

Next, we show Pareto efficiency. To find a contradiction, suppose for some preference profile, $R \in \mathcal{R}$, a matching plan $\Pi(R)$ given by an SD futures mechanism is not Pareto efficient. For notational simplicity, let $\mu := \Pi(R)$. Then, there exists a matching plan, ν, that Pareto dominates μ in R. Thus, $\forall a \in A$, $\nu_a R_a \mu_a$ and $\exists b \in A$ such that $\nu_b P_b \mu_b$. Since $A \equiv A(1) \cup (\cup_{t\geq 2} N(t))$, agent b is either in $A(1)$ or in $N(t)$ for some $t \geq 2$. Suppose $\forall a \in A(1)$ $\mu_a R_a \nu_a$.

Otherwise, the proof is similar to the following and is therefore omitted. Take the smallest $\hat{t} \geq 2$ such that $\exists b \in N(\hat{t})$ with $\nu_b \, P_b \, \mu_b$. It follows from strict preferences that $\forall a \in A(1)$, $\nu_a = \mu_a$ and $\forall t$ with $2 \leq t \leq \hat{t} - 1$, $\forall a \in N(t)$, $\nu_a = \mu_a$. Consider an agent $b \in N(\hat{t})$ who has the highest order in $f_{N(\hat{t})}$ such that $\nu_b P_b \mu_b$. Then, it follows from strict preferences that for each agent a who has higher order than b, $\nu_a = \mu_a$. Thus, in the SD futures mechanism, when it is agent b's turn to choose, two assignments, ν_b and μ_b, are still available. Thus, since agent b chooses μ_b in the SD futures mechanism, $\mu_b R_b \nu_b$. This is a contradiction.

Since the SD futures mechanism is proved to be Pareto efficient, it follows from Impossibility Theorem 1 that it is not acceptable.

□

7 Summary

We summarize some of our results in the tables below.

Table 4: Properties of dynamic mechanisms under "general" preferences

	AC	SP	PE
General SD spot mechanism		✓	
Constant SD spot mechanism favoring existing tenants		✓	✓
SD spot mechanism favoring newcomers		✓	
TTC spot mechanism	✓		
SD futures mechanism		✓	✓

Note: AC stands for acceptability, SP stands for strategy-proofness, and PE stands for Pareto efficiency. The mark "✓" in a cell indicates that a corresponding dynamic mechanism in the first column satisfies the corresponding properties in the first row. On the other hand, a blank cell indicates that the dynamic mechanism does not satisfy the property. Moreover, ✓* indicates that the spot mechanism is acceptable for a problem without endowments (See Proposition 2 in the Appendix). ✓** shows that it is SP (PE) for a problem without endowments and SP (PE) among all agents except initial existing tenants for a problem with endowments.

These results verify the desirableness of seniority-based mechanisms.

Table 5: Properties of dynamic mechanisms under "time-invariant" preferences

	AC	SP	PE
General SD spot mechanism		✓	
Constant SD spot mechanism favoring existing tenants	✓*	✓	✓
SD spot mechanism favoring newcomers		✓	
General TTC spot mechanism	✓		
Constant TTC spot mechanism favoring existing tenants	✓	✓**	✓**
TTC spot mechanism favoring newcomers	✓		
SD futures mechanism		✓	✓

Appendix

Proposition 2. *Consider a dynamic problem with time-invariant preferences and without endowments. A constant SD spot mechanism favoring existing tenants is acceptable.*

Proof. Suppose that an agent, a, obtains some house h at some period t. Since agents before agent a do not prefer the house h in period t, in the next period, it follows from time-invariant preferences that agents before agent a do not obtain this house, and thus agent a has an option of getting house h or one of the remaining houses. Hence, agent a is weakly better off as time goes on. □

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