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Faulty Nash Implementation in Exchange Economies with Single-peaked Preferences

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Abstract. In this paper, we reexamine Eliaz's results (2002) of fault tolerant implementation on one hand and we extend theorems 1 and 2 of Doghmi and Ziad (2008 *a*) to bounded rationality environments, on the other. We identify weak versions of the k-no veto power condition, in conjunction with unanimity and variants of k-monotonicity, are sufficient for implementability in k- Fault Tolerant Nash equilibria (k-FTNE). In addition, these new conditions are stable by intersection which makes it possible to check directly the $k-FTNE$ implementability of the social choice correspondences. We apply these results to exchange economies with single -peaked preferences, to finite allocation problems, and to equilibrium theory. Firstly, we note that our conditions are satisfied by all monotonic solutions contrary to Eliaz's results (2002). Secondly, in exchange economy when preferences are single-peaked, the k-monotonicity is sufficient for the $k-FTNE$ implementation for the *correspondences* and both necessary and sufficient for the *functions*. However, the results are negatives for the no-monotonic solutions.

Keywords: Faulty Nash implementation; Bounded rationality; Exchange economies; Single-peaked preferences

JEL classification: C72; D71

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1 Introduction

Implementation theory studies the problem of a planner who faces a certain number of agents and a set of options. Each agent has preferences over various options. A difficulty arises in the asymmetry of information between planner and agents. This asymmetric information comes because the planner does not know the exact preferences of agents. If for example, the options are public goods, the agents will state false preferences in order to participate with lower cost, and once the public property is constructed, they can use it like everyone. So that the agents reveal their true preferences, the planner will implement a mechanism (non-cooperative game) where the strategies of players depend essentially on the preference profiles and the options (or alternatives) set. A social choice correspondence (SCC) is implementable in a given solution concept if the payment with this solution of the game corresponds to the socially desired alternative and vice versa. The planner, thus, hopes via this game to get the agents to reveal their sincere preferences.

In a complete information environment where the solution concept is Nash equilibrium and the agents are rational in their behavior, Maskin (1977, 1999) was the first to be interested in this question. He showed that a SCC which is Nash implementable must satisfy a property now known as *Maskin monotonicity.* This property stipulates that if an option is socially chosen with a preference profile R, and if all other options ranked below it (in the large sense) remain below it for a new profile R', then this option must be chosen with the preference profile R'. For sufficiency, Maskin proposes an additional property called *no veto power*, which stipulates that if an option is better than all other options for all players except at most one, this option must be socially selected. However, this latter property is sufficient but not necessary for Nash implementation. Thus, in the literature of implementation theory, several authors have suggested replacements for no veto power property (Moore and Repullo 1990; Sjöström 1991; Danilov 1992; Yamato 1992; Ziad 1997,1998; Bochet 2007; Benoit and Ok 2006,2008; Doghmi and Ziad 2008 *a*, 2008 *b*; and Zhou 2008).

Moore and repullo (1990) gave a full characterization (i.e., necessary and sufficient conditions) for an implementable SCCs. Unfortunately, these conditions depend on *a priori* unknown sets. Sjöström (1991) proposed an algorithm to determine these unknown sets and then to apply the characterization of Moore and Repullo. The weakness of this algorithm is that it is not operational when the SCC is not implementable (in this case, it does not produce an answer). Ziad (1997,1998) cured this defect by generalizing the concept of monotonicity introduced initially by Maskin. Danilov (1992) and Yamato (1992) proposed an elegant sufficient condition called *strong monotonicity*.

Thomson (1990) tried to apply these various conditions to models of exchange economies with single-peaked preferences. Thomson showed that the Maskin theorem applies only to the Pareto correspondence. Also, he demonstrated that the no envy correspondence, the individually rational correspondence from equal division, and the intersection of these two correspondences satisfy strong monotonicity. Then they are Nash implementable as well. However, Thomson illustrated that the intersections of some strong monotonic correspondences do not satisfy strong monotonicity e.g., the intersection of the individually rational correspondence from equal division and the Pareto correspondence, or the intersection of the no envy correspondence and Pareto correspondence. Strong monotonicity is unfortunately unstable by intersection.

Doghmi and Ziad (2008 *a*) identified new sufficient conditions called strict monotonicity, strict weak no veto power and weak no veto power. They showed that any SCC satisfying either the conjunction of strict monotonicity with strict weak no veto power and unanimity or the conjunction of Maskin monotonicity with weak no veto power and unanimity is Nash implementation. These results extend Maskin's theorem. In addition, intersections of SCCs satisfying these conditions also satisfy the same conditions. This stability property is an extremely useful tool in identifying Nash implementable SCCs. Doghmi and Ziad (2008 *b*) provided applications of these new results to solutions of the problems of fair division in exchange economies with single-peaked preferences, and to finite allocation problems .

In exchange economies with single-peaked preferences, Doghmi and Ziad (2008 *b*) gave a full characterization by showing that Maskin monotonicity, alone, is necessary and sufficient for Nash implementability. Thus, in this setting, the problems of identifying Nash implementable SCCs have finally been solved. Also, Doghmi and Ziad (2008 *b*) gave a generalization for implementability of the solutions of the problems of finite allocations. But the question is: what about the implementability of these solutions in bounded rationality environments?

In standard implementation theory, the solution concept analyzes *rational behavior* of players. Each player can choose its best strategy. However, this literature of full rationality has been criticized by several authors (Hurwicz, 1986; Moore, 1992; Sjostrom, 1993; Segal, 1999; Cabrales, 2000; and Cabrales and Ponti, 2000). The question is how to find a solution when there are slight deviations from full rationality. Part of the answer, in complete information environments, was given by Eliaz (2002) (see also Doghmi and Ziad, 2007 and Matsushima, 2008 for Bayesian environments). In his model, Eliaz assumes that there is a minority of k players who are "faulty" in the sense that they do not understand the rules of the game, or they make mistakes. The planner and non-faulty players know only that there exist at most k faulty players in the population. They know neither their identity, nor their exact number, nor their behaviors. Eliaz defined a new solution concept called k-Fault Tolerant Nash Equilibrium ($k-FTNE$). This solution concept requires that a player responds in an optimal manner to the non-faulty players, independently of the identity and strategies of the faulty players. In a society of at least three players, Eliaz (2002) showed that any SCC that satisfies a version of monotonicity (stronger than Maskin monotonicity, called k-monotonicity) and the no veto power condition is $k - FTNE$ implementable when faulty players are in a minority. He showed also that any $k-FTNE$ implementable SCC must satisfy weak k-monotonicity.

Thus, Eliaz (2002) uses the no veto power condition. This condition is not satisfied by several applications. We consider the following domains.

(1) *Exchange economies with single peaked preferences.* In this domain, only the Pareto correspondence (P) can (probably) be implemented by Eliaz's result (2002), but the no-envy correspondence(NE), the individually rational correspondence from equal division (I_{ed}), the core correspondence from equal division, the group no-envy correspondence, the ($P \cap NE$) correspondence, the ($P \cap I_{ed}$) correspondence, and the ($NE \cap I_{ed}$) correspondence cannot.

(2) *Finite allocation problems.* In this domain, although the weak Pareto correspondence can (probably) be implemented in $k-FTNE$ by Eliaz's conditions, the weak Core correspondence cannot.

(3) *Equilibrium theory.* For this setting, in pure exchange economies, the constrained Walrasian equilibrium is implementable in $k - FTNE$ by Eliaz's conditions, but in the general case of exchange economies with production and with possibly satiated preferences, the constrained Walrasian equilibrium with slack is not.

With two examples, Eliaz (2002) shows that there is no logical relationship between the necessary condition for Nash implementation and the necessary condition for faulty Nash implementation. For *non Nash implementable correspondences*, this is encouraging, they may be faulty Nash implementability. For this, we consider the following environments.

(4) *The no-monotonic solutions in exchange economies with single peaked preferences.* The Proportional solution, the Symmetrically proportional solution, the Equal-distance solution, and the Equal-distance solution do not satisfy Maskin monotonicity. Therefore, it are not Nash implementable. Can these solutions be implemented in $k - FTNE$?

(5) *The no-monotonic solutions in finite allocation.* The strong Pareto correspondence and the strong Core correspondence do not satisfy Maskin monotonicity . Therefore, they are not Nash implementable. Can these correspondences be implemented in $k - FTNE$?

To answer these questions, we extend theorems 1 and 2 of Doghmi and Ziad (2008 a) by reexamining Eliaz's results of fault tolerant implementation. We give new sufficient conditions called *strict k-monotonicity*, *strict weak k-no veto power*, and *weak k-no veto power*. We propose two combinations. The first combination contains strict k-monotonicity, strict weak k-no veto power, and unanimity. In the second combination, we consider the conjunction of Maskin monotonicity with weak k-no veto power, and unanimity. We demonstrate that if a social choice correspondence satisfies either of these new combinations, it can be implemented in $k - FTNE$.

The rest of this paper is organized as follows. In Section 2, we introduce notations and definitions. In Section 3, we state and prove our main results. In sections 4, 5 and 6, we give several applications of our results with respect to domains of exchange economies with single peaked preferences, to finite allocation problems, and to equilibrium theory. We conclude by remarks.

2 Notations and definitions

Let A be a set of alternatives, and let $N = \{1, ..., n\}$ be a finite set of individuals, with generic element i. Each individual i is characterized by a preference relation R_i defined over A, which is a complete, transitive, and reflexive relation in some class $\Re_i$ of admissible preference relations. Let $\Re = \Pi_{i \in N} \Re_i$. An element $R = (R_1, R_2, ..., R_n) \in \Re$ is a preference profile. The relation R_i indicates the individual's i preference. For $a, b \in A$, the notation aR_ib means that the individual i prefers weakly a to b. The asymmetrical and symmetrical parts of R_i are noted respectively by P_i and I_i.

A social choice correspondence (SCC) F is a function from $\Re$ into $2^A \setminus \emptyset$, that associates with every R a nonempty subset of A. For all $R_i \in \Re_i$ and all $a \in A$, the lower contour set for agent i at alternative a is noted by:

$L(a, R_i) = \{b \in A \mid aR_ib\}$. The strict lower contour set and the indifference lower contour set are noted respectively by $LS(a, R_i) = \{b \in A \mid aP_ib\}$ and $LI(a, R_i) = \{b \in A \mid a \sim_i b\}$.
A mechanism (or form game) is given by $\Gamma = (S, g)$ where $S = \Pi_{i \in N} S_i$; S_i denotes the strategy set of the agent i and g is a function from S to A. The elements of S are denoted by $s = (s_1, s_2, ..., s_n) = (s_i, s_{-i})$, where $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$. When $s \in S$ and $b_i \in S_i$, $(b_i, s_{-i}) = (s_1, ..., s_{i-i}, b_i, s_{i+1}, ..., s_n)$ is obtained after replacing s_i by b_i, and $g(S_i, s_{-i})$ is the set of results which agent i can obtain when the other agents choose s_{-i} from $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.
A Nash equilibrium of the game (Γ, R) is a vector of strategies $s \in S$ such that for any i, $g(s)R_ig(b_i, s_{-i})$ for all $b_i \in S_i$, i.e. when the other players choose s_{-i}, the player i cannot deviate from s_i. Given $N(g, R, S)$ the set of Nash equilibria of the game (Γ, R), a mechanism $\Gamma = (S, g)$ implements a SCC F in Nash equilibria if for all $R \in \Re$, $F(R) = g(N(g, R, S))$.
We say that a SCC F is implementable in Nash equilibria if there is a mechanism which implements it in these equilibria.

A k-Fault Tolerant Nash Equilibrium ($k-FTNE$) of the game (Γ, R) is a vector of strategies $s^* \in S$ such that for any i, $g(s_i^*, s^*_{N \setminus M \cup \{i\}}, s_M)R_ig(b_i, s^*_{N \setminus M \cup \{i\}}, s_M)$ for all $b_i \in S_i$, for all $s_M \in S_M$ and for all $M \subseteq N$ such that $\mid M \mid \leq k$, i.e. when the other non-faulty players choose $s^*_{N \setminus M \cup \{i\}}$ regardless of the strategies of the faulty players, the non-faulty player i cannot deviate from s_i^*. The difference $d(s, s')$ between any pair of strategies s and s', such that $d(s, s') = \mid \{i \in N : s_i \neq s'_i\} \mid$, measures the number of players who do not choose the same strategies in both profiles. Given $B(s, k)$ the set of profiles that are different from s by at most k strategies such that for any profile of strategies $s \in S$: $B(s, k) = \{s' \in S : d(s, s') \leq k\}$. Given $N^k(g, R, S)$ the set of $k-FTNE$ of the game (Γ, R), a mechanism $\Gamma = (S, g)$ implements a SCC F in $k-FTNE$ if for all $R \in \Re$, $F(R) = g(N^k(g, R, S))$ and $g(B(s^*, k)) \subseteq F(R)$ for all $s^* \in N^k(g, R, S)$.
We say that a SCC F is implementable in $k-FTNE$ if there is a mechanism which implements it in this notion of equilibrium.

Now, we introduce the fundamental conditions in classical Nash implementation theory. Maskin (1977/1999) showed that any Nash implementable SCC must satisfies the following property:

Definition 1 *(Maskin Monotonicity). A SCC F satisfies monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $L(a, R_i) \subseteq L(a, R'_i)$, then $a \in F(R')$.*

For sufficiency, Maskin (1977/1999) gave the following additional property:

Definition 2 *(No-veto power). A SCC F satisfies no veto power if for i, $R \in \Re$, and $a \in A$, if $L(a, R_j) = A$ for all $j \in N \backslash \{i\}$, then $a \in F(R)$.*

Next, we introduce the Doghmi and Ziad's conditions (2008) which are used to reexamine Maskin's theorem (1977, 1999):

Definition 3 *(strict monotonicity). A SCC F satisfies strict monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $LS(a, R_i) \cup \{a\} \subseteq L(a, R'_i)$, then $a \in F(R')$.*

Definition 4 *(Versions of weak no veto power)*
***i) strict weak no veto power**. A SCC F satisfies strict weak no veto power if for i, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$, $b \in LS(a, R_i) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, then $b \in F(R')$.*
***ii) weak no veto power**. A SCC F satisfies weak no veto power if for i, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$, $b \in L(a, R_i) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, then $b \in F(R')$.*

Definition 5 *(Unanimity). A SCC F satisfies unanimity if for any $a \in A$ and any $R \in \Re$, if for any $i \in N$, $L(a, R_i) = A$, then $a \in F(R)$.*

Finally, we introduce the fundamentals conditions in bounded rationality environment. In this setting, Eliaz (2002) identified the following conditions:

Definition 6 *(k-monotonicity). A SCC F satisfies k-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $M \subseteq N$ with $\mid M \mid \geq k+1$ and for some $i \in M$, $L(a, R_i) \subseteq L(a, R'_j)$ for all $j \in M$, then $a \in F(R')$.*

Definition 7 *(weak k-monotonicity). A SCC F satisfies weak k-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $M \subseteq N$ with $\mid M \mid \geq k+1$ and for some $i \in M$, there is an outcome $a^i \in F(R)$ satisfying $L(a^i, R_i) \subseteq L(a^j, R'_j)$ for all $j \in M$, then $a \in F(R')$.*

Eliaz (2002) showed that any SCC satisfying k-monotonicity and no veto power conditions can be implemented in $k - FTNE$, and all $k - FTNE$ implementable SCC's are weakly k-monotonic.

3 New sufficient conditions for faulty Nash implementation

In this section, we state and prove our main results. We start by introducing the following condition: it says that if an alternative is selected for some profile of preferences R, and for all subsets M of at least $k+1$ players, if by *at least one* of them, all subset contains the chosen alternative and all other alternatives ranked strictly below it remain below it (in large sense) for *all* players of M for a new profile R', then this alternative must be chosen for the new profile R'.

Definition 8 *(strict k-monotonicity). A SCC F satisfies strict k-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $M \subseteq N$ with $\mid M \mid \geq k+1$ and for some $i \in M$, $LS(a, R_i) \cup \{a\} \subseteq L(a, R'_j)$ for all $j \in M$, then $a \in F(R')$.*

Remarks.

- strict k-monotonicity implies k-monotonicity. They become equivalent, if the preferences are strict.
- strict k-monotonicity implies weak k-monotonicity.
- strict k-monotonicity implies Maskin monotonicity.
- if $k = 0$ then strict k-monotonicity reduce to Doghmi and Ziad's condition (2008) of strict monotonicity.

Next, we introduce two weak versions of k-no veto pwer.[1]

Definition 9 *(strict weak k-no veto power). A SCC F satisfies strict weak k-no veto power if for some $M \subseteq N$ with $\mid M \mid \leq k+1$, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$ and for some $i \in M$, $b \in LS(a, R_i) \subseteq L(b, R'_j)$ for all $j \in M$ and $L(b, R'_j) = A$ for all $j \in N \backslash M$, then $b \in F(R')$.*

Definition 10 *(weak k-no veto power). A SCC F satisfies weak k-no veto power if for some $M \subseteq N$ with $\mid M \mid \leq k+1$, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$ and for some $i \in M$, $b \in L(a, R_i) \subseteq L(b, R'_j)$ for all $j \in M$ and $L(b, R'_j) = A$ for all $j \in N \backslash M$, then $b \in F(R')$.*

[1] A SCC F satisfies k-no veto power if for some $M \subseteq N$ with $\mid M \mid \leq k+1$, $R \in \Re$, and $a \in A$, if $L(a, R_j) = A$ for all $j \in N \backslash M$, then $a \in F(R)$.

If $k = 0$ then strict weak k-no veto and weak k-no veto power power reduce to Doghmi and Ziad's conditions (2008) of strict weak no veto power and weak no veto power.

Finally, to characterize the $k - FTNE$ implementable SCC's, we propose two combinations. In the first combination, we consider the conjunction of strict k-monotonicity, strict weak k-no veto power and unanimity. In the second combination, we bind Maskin monotonicity with weak k-no veto power and unanimity conditions. We show that any SCC satisfying either of these two combinations can be implemented in $k - FTNE$, as the following theorems illustrate.

Theorem 1 *Let $n \geq 3$ and $k + 1 < \frac{n}{2}$. If a SCC F satisfies strict k-monotonicity, strict weak k-no veto power and unanimity, then F can be implemented in $k - FTNE$.*

Theorem 2 *Let $n \geq 3$ and $k + 1 < \frac{n}{2}$. If a SCC F satisfies k-monotonicity, weak k-no veto power and unanimity, then F can be implemented in $k - FTNE$.*

Because the proofs of theorems 1 and 2 follow the same arguments, we give just the proof of theorem 1.

Proof of theorem 1. For this proof, we utilize the same fault tolerant mechanism used by Eliaz (2002). Let $\Gamma = (S, g)$ be a mechanism which is defined as follows: For each $i \in N$, let $S_i = \Re \times A \times \mathbb{N}$, where $\mathbb{N}$ consists of the nonnegative integers. The generic element of strategic space S_i is noted by: $s_i = (R_i, a_i, m_i)$. Each agent announces a preference profile, an optimal alternative for this profile and a nonnegative integer. The function g is defined as follows:

Rule 1: If at least $n - k$ players announce (R, a, m) such that $a \in F(R)$, then the outcome is a.

Rule 2: If exactly $n - k - 1$ players announce (R, a, m) such that $a \in F(R)$, then the outcome is a, *unless* all of the remaining $k + 1$ players agree on $(R', b, .)$and $b \in LS(a, R_i) \neq \emptyset$ for everyone of them, in which case the outcome is b.

Rule 3: In any other situation, $g(s) = a_{i^*}$, where i^* is the index of the player of which the number m_{i^*} is largest (in case of a tie the identity of i^* is immaterial).

Before proving the result, let precise that for a profile of strategies, if two or more strategies are identical, we suppose for simplicity and without any restriction of generality that these strategies are the first one. Also to simplify the proof, for a profile s^* containing more that $n-k$ identical strategies, we write $s^* = (s_1^*, s_2^*, ..., s_{n-k}^*, s_{n-k+1}, ..., s_n)$, else we write $s^* = (s_1^*, s_2^*, ..., s_{n-k'}^*, s_{n-k'+1}, ..., s_n)$, with $k' > k$. Let us show that for all $R \in \Re$, $F(R) = g(N^k(g, R, S))$ and $g(B(s^*, k)) \subseteq F(R)$ for all $s^* \in N^k(g, R, S)$. The proof contains two steps:

Step 1. For all $R \in \Re$, $F(R) \subseteq g(N^k(g, R, S))$.

Let $R \in \Re$ and $a \in F(R)$. For each $i \in N$, let $s_i = (R, a, 1)$. Then, by rule 1, $g(s) = a$ and for all $s' \in B(s, k)$, $g(s') = a$. We want to show that $s \in N^k(g, R, S))$. Let us choose any individual i and any deviation from s' by the strategy $\widetilde{s}_i$ such that $\widetilde{s}_i = (\widetilde{R}, a_i, \widetilde{m})$. If $a_i \in LS(a, R_i) \neq \emptyset$, then, by rule 2, $g(\widetilde{s}_i, s'_{-i}) = a_i$. But, since $LS(a, R_i) \subseteq L(a, R_i)$, then $g(s) R_i g(\widetilde{s}_i, s'_{-i})$, thus $s \in N^k(g, R, S))$. If $a_i \notin LS(a, R_i)$, then $g(s) = g(\widetilde{s}_i, s'_{-i})$, thus $s \in N^k(g, R, S))$.

Step 2. For all $R \in \Re$, $g(N^k(g, R, S)) \subseteq F(R)$ and $g(B(s^*, k)) \subseteq F(R)$ for all $s^* \in N^k(g, R, S)$.

Let $s^* \in N^k(g, R, S)$. Let us show that $g(s^*) \in F(R)$ and $\forall s' \in B(s^*, k)$, $g(s') \in F(R)$. For that, we will study the various possibilities of writing the profile of strategies s^*.

Substep 2.1. For all $R \in \Re$, $g(N^k(g, R, S)) \subseteq F(R)$.

Case a: $s^* = (s_1^*, s_2^*, ..., s_{n-k}^*, s_{n-k+1}, ..., s_n)$ and $s' = (s_1^*, s_2^*, ..., s_{n-k}^*, s'_{n-k+1}, ..., s'_n)$. Suppose there exists $(R', a, m) \in \Re \times A \times \mathbb{N}$, with $a \in F(R')$, such that s^* is defined by $s_i^* = (R', a, m)$ for any $i \in N$. Then, by rule 1, $g(s^*) = a$ and $\forall s' \in B(s^*, k)$, $g(s') = a$.

Let H^k be the subset of k players deviating from s^* and let us take any $i \in N \setminus H^k$. Let M be the union of the all subset H^k of k players and the various players $i \in N \setminus H^k$, i.e., $M \equiv H^k \cup i$, with $\mid M \mid \geq k+1$. Since, each subset M contains at least one non-faulty player i',[2] we take any $b \in LS(a, R'_{i'}) \cup \{a\}$. Let $\widetilde{s}_{i'} = (R', b, m')$ a deviation for the non-faulty player i'. For a profile $s' \in S$, the k players of the subset H^k play (R', b, m'). Then, by rule 2, $g(\widetilde{s}_{i'}, s^*_{N \setminus M}, s'_{M \setminus \{i'\}}) = b$ for all $j \in M$. Since $s^* \in N^k(g, R, S)$, $a = g(s^*) R_j g(\widetilde{s}_{i'}, s^*_{N \setminus M}, s'_{M \setminus \{i'\}}) = b$ for all $j \in M$. Therefore, $LS(a, R'_{i'}) \cup$

[2]To clarify this point, we consider this example. Let $N = \{1, 2, 3, ..., 7\}$ and $k = 2$. Without loss of generality, we fix the index 1 and 2 for the faulty players such that $H^2 \equiv \{1, 2\}$. In this case, for all $i \in N \setminus H^2 \equiv \{3, 4, ..., 7\}$, the subset M with $\mid M \mid \geq 3$ takes the following forms: $M \equiv \{1, 2, 3\}, M \equiv \{1, 2, 3, 4\}, ..., M \equiv N$. Thus, subset M contains at least one non-faulty player of the index 3.

$\{a\} \subseteq L(a, R_j)$ for all $j \in M$. By strict k-monotonicity, $a \in F(R)$.

Case b: $s^* = (s_1^*, s_2^*, ..., s_{n-k}^*, s_{n-k+1}, ..., s_n)$ and $s' = (s_1^*, s_2^*, ..., s_{n-k}^*, s'_{n-k+1}, ..., s'_n)$. Assume there is a subset H^k of k players in N, a player τ in $N \setminus H^k$, we let $M \equiv H^k \cup \{\tau\}$, with $| M |= k+1$. Let $R' \in \Re$ and $a \in A$ such that $a \in F(R')$. For all $j \in N \setminus M$, $s_j^* = (R', a, m)$ and $s_M = (s_\tau^*, s_{H^k}) = (R'', a', m') \neq s_j^*$. In the profile s^* there are exactly $n-k-1$ players announce (R', a, m) and the remaining $k+1$ players agree on (R'', a', m'), in this case,

$$g(s^*) = \begin{cases} a' & \text{if} \quad a_i \in LS(a, R_i) \neq \emptyset \\ a & \text{otherwise.} \end{cases}$$

There are two subcases:

Subcase b_1***:*** If $g(s^*) = a'$,
By definition $a' \in LS(a, R'_i) \neq \emptyset$ for all $i \in M$. Since the subset M contains at least one non-faulty player τ, we take any $b \in LS(a, R'_\tau) \neq \emptyset$. Let $\widetilde{s}_\tau$ be a deviation such that $\widetilde{s}_\tau = (\widetilde{R}, b, \widetilde{m})$ and let $\widetilde{s}_{H^k}$ be a deviation by the subset H^k of k players such that $\widetilde{s}_{H^k} = (\widetilde{R}, b, \widetilde{m})$. Then, by rule 2, $g(\widetilde{s}_\tau, s^*_{N\setminus M}, \widetilde{s}_{M\setminus\{\tau\}}) = b$ for all $j \in M$. But, since $s^* \in N^k(g, R, S)$, $b \in L(a', R_j)$ for all $j \in M$. Hence, we have $a' \in LS(a, R'_\tau) \subseteq L(a', R_j)$ for all $j \in M$. (1)

Next, because $n \geq 3$ and $k+1 < \frac{n}{2}$, for any other deviation $j \in N \setminus M$ and any $b \in A$, let $\widetilde{s}_j = (\widetilde{R}, b, \widetilde{m})$ a deviation. In this case the subset H^k of k players will play the same strategy profile $\widetilde{s}_{H^k} = (\widetilde{R}, b, \widetilde{m})$ as that of player j, $\widetilde{m}$ is the unique greatest integer in the profile $(\widetilde{s}_j, s^*_{N\setminus H^k\cup j}, \widetilde{s}_{H^k})$. By rule 2, $g(\widetilde{s}_j, s^*_{N\setminus H^k\cup j}, \widetilde{s}_{H^k}) = b$. Since $s^* \in N^k(g, R, S)$, we have $a' = g(s) R_i g(\widetilde{s}_j, s^*_{N\setminus H^k\cup j}, \widetilde{s}_{H^k}) = b$. Therefore, $A \subseteq L(a_i, R_j)$ for all $j \in N \setminus M$. (2)

From (1), (2) and by k-strict weak no veto power, we have $a' \in F(R)$.

Subcase b_2***:*** If $g(s^*) = a$,
By the same reasoning used in case a, we obtain by strict k-monotonicity that $a \in F(R)$.

Case c: $s^* = (s_1^*, s_2^*, ..., s_{n-k}^*, s_{n-k+1}, ..., s_n)$ and $s' = (s_1^*, s_2^*, ..., s_{n-k}^*, s'_{n-k+1}, ..., s'_n)$: $\forall i \in N$, $s_i^* = (R', a, m)$ with $a \notin F(R')$, $g(s^*) = a$. Let $b \in A$ and $i \in N$, $\widetilde{s}_i = (R', b, m')$, where $m' > m$, then, $g(\widetilde{s}_i, s^*_{-i}) = b$. As $s^* \in N^k(g, R, S)$, then, $a = g(s^*) R_i g(\widetilde{s}_i, s^*_{-i}) = b$. Therefore, $A \subseteq L(a, R_i)$ for all $i \in N$. Since the SCC F satisfies unanimity then $a \in F(R)$.

Case d: The last case is the opposite of the previous cases,
$s^* = (s_1^*, s_2^*, ..., s_{n-k'}^*, s_{n-k'+1}, ..., s_n)$ with $k' > k$. We obtain $g(s^*) = a_l$: m_l is the maximum of the integers m. Let $b \in A$, and $\widetilde{s}_i = (R', b, m_l + 1)$ a deviation. Therefore, $g(\widetilde{s}_i, s^*_{-i}) = b$. As $s^* \in N(g, R, S)$, then,

$g(s^*)R_ig(\widetilde{s}_i, s^*_{-i}) = b$. Thus, $A \subseteq L(g(s^*), R_i)$ for all $i \in N$. By unanimity $g(s^*) \in F(R)$.

Substep 2.2. For all $R \in \Re$, $g(B(s^*, k)) \subseteq F(R)$ for all $s^* \in N^k(g, R, S)$. Let $s^* \in N^k(g, R, S)$ and suppose that $g(s^*) = a$ and let $s' \in B(s^*, k)$, if $g(s') = a$ we have no thing to prove. In the following we suppose that $g(s') \neq a$

Let $s^* \in N^k(g, R, S)$ and $s' \in B(s^*, k)$ be profiles of strategies. Assume that $g(s')$ is defined by rule 2. In s' there are exactly $n - k - 1$ players who agree on (R', a, m) and the remaining agree on (R'', a', m')

$$g(s') = \begin{cases} a' & \text{if} \quad a' \in LS(a, R_i) \neq \emptyset \\ a & \text{otherwise.} \end{cases}$$

By the same reasoning used in subcases b_1 and b_2, we obtain that $g(s') \in F(R)$.

Secondly when $g(s')$ is determined by Rule 3. Let $s'_i = (R', c, \widetilde{m})$ a strategies profile where a subset M (with cardinality less or equal k) of players deviate from s^*, with $\widetilde{m} > m$. Let $\widetilde{s} = (s^*_1, s^*_2, ..., s^*_{n-k}, \widetilde{s}_{n-k+1}, ..., \widetilde{s}_n)$ and $\widetilde{s}_M = (\widetilde{R}, d, \widetilde{m}')$, with $\widetilde{m}' > \widetilde{m}$. Then, $\widetilde{s} \in B(s^*, k)$, by Rule 3, $g(\widetilde{s}) = d$. Let $\widehat{s} = (s^*_1, s^*_2, ..., \widehat{s}_j, ..., s^*_{n-k}, \widetilde{s}_{n-k+1}, ..., \widetilde{s}_n)$ a strategies profile in which $j \in N \setminus M$ announces $\widehat{s}_j = (\widehat{R}, e, \widehat{m})$ with $\widehat{m} > \widetilde{m}'$ and $\widehat{s}_{-j} = \widetilde{s}_{-j}$. Then, by Rule 3, $g(\widehat{s}) = e$. Therefore, each player has some beliefs on the possibility of deviation of faulty players. He obtains his most preferred outcome by announcing the largest integer when others players play their strategies. Since $s^* \in N^k(g, R, S)$, it must that $a = g(s^*) \sim_i b$ for all $i \in N$ and for all $b \in A$. Thus, for all $s' \in B(s^*, k)$, $g(s') \sim_i b$ for all $i \in N$ and for all $b \in A$. Therefore $A \subseteq L(g(s'), R_i)$. Since the SCC F satisfies unanimity then $g(s') \in F(R)$. Q.E.D.

4 Applications in exchange economies with Single-peaked preferences

Let $N = \{1, ..., n\}$ be a set of n agents which must share a quantity $\Omega \in \mathbb{R}_{++}$ of a certain perfectly divisible good. The preference of each agent $i \in N$ is represented by a continuous and single-peaked preference relation R_i over $[0, \Omega]$ (the asymmetrical part is written P_i and the symmetrical part I_i). For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ mean that, for the agent i, to consume a share x_i is as good as to consume the quantity y_i.

Definition 11 *(Single-peaked preferences)*
A preference relation R_i is single peaked if there is a number $p(R_i) \in [0, \Omega]$ such that for all $x_i \in [0, \Omega]$ if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$. We call $p(R_i)$ the peak of R_i.

Let $\mathscr{D} \subseteq \Re$ be the class of all such preference relations. For $R \in \mathscr{D}$, let $p(R) = (p(R_1), ..., p(R_n))$ be the profile of peaks (or of preferred consumptions). A single peaked preference relation $R_i \in \mathscr{D}_i$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the peak which is indifferent to x_i (if it exists), else, it is 0 or Ω. Formally, if $x_i \leq p(R_i)$, then, $r_i(x_i) \geq p(R_i)$ and $x_i I_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \geq p(R_i)$, then, $r_i(x_i) \leq p(R_i)$ and $x_i I_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise.

A feasible allocation for the economy (R, Ω) is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. We note that the feasible allocations set is $X = [0, \Omega] \times [0, \Omega] \times ... \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. For the set $L(x, R_i) = X$, $x R_i y$ for all $y \in X$ implies that $x_i R_i y_i$.

Lemma 1 *(Doghmi and Ziad (2008 b))*
Let $R, R' \in \mathscr{D}$ and $x, y \in X$. If preferences are single-peaked, $y_i \in LS(x_i, R_i)$, and $LS(x_i, R_i) \subseteq L(y_i, R'_i)$, then $L(y_i, R'_i) = [0, \Omega]$.

In exchange economies with single peaked preferences, we have the following proposition.

Proposition 1 *: In exchange economies with single peaked preferences, strict k-monotonicity becomes equivalent to k-monotonicity.*

Proof. This proof is omitted as it follows the same argument as proof of Proposition 4 in Doghmi and Ziad (2008b).

Theorem 3 *Let $n \geq 3$ and $k + 1 < n/2$. Let F be a SCC (respectively let $f : \Re \to A$ be a social choice function) satisfying i) unanimity, ii) $(P(R_1), P(R_2), ..., P(R_n)) \in F(R)$, then F can be implemented in k-FTNE if F satisfies k-monotonicity (respectively f can be implemented in k-FTNE if and only if f satisfies k-monotonicity).*

Proof. By Lemma 1 and unanimity, the strict weak k-no veto power condition always holds independently of the solution. The condition ii) implies strict weak k-no veto power for F. By Proposition 1, strict k-monotonicity is equivalent to k-monotonicity. For a social choice function, the k-monotonicity condition becomes equivalent to weak k-monotonicity which is necessary condition for $k-FTNE$ implementability. Q.E.D.

For $k = 0$, the theorem 3 has as a role to close the gap in Maskin theorem in exchange economies with single-peaked preferences. In this environment, the Maskin monotonicity condition is, alone, necessary and sufficient for Nash implementation.

a) Pareto correspondence, P

The Pareto correspondence is the solution which associates each economy with its feasible allocation set such that there does not exist any other feasible allocation that all agents prefer weakly and at least one prefers strictly.

$P(R) = \{x \in X : \nexists x' \in X$ such that for all $i \in N$, $x'_i R_i x_i$, and for some $i \in N$, $x'_i P_i x_i\}$.

Let $x \in P(R)$. It is easy to check that if $\sum_{i \in N} p(R_i) \geq \Omega$, for all $i \in N$, then $x_i \leq p(R_i)$ and that if $\sum_{i \in N} p(R_i) \leq \Omega$, for all $i \in N$, then $x_i \geq p(R_i)$.[3]

Proposition 2 *: Let $0 \leq k < n$. The Pareto correspondence satisfies strict k-monotonicity.*

Proof. Let $R, R' \in \mathscr{D}$ and $x, y \in X$. Let $x \in P(R)$. Suppose that $\sum_{i \in N} p(R_i) \geq \Omega$ (similar statements can be proved for $\sum_{i \in N} p(R_i) \leq \Omega$). Then, $\forall i \in N$, $x_i \leq p(R_i)$. Suppose that the Pareto correspondence does not satisfies strict k-monotonicity. Therefore, for all $M \subseteq N$, with $\mid M \mid \geq k+1$ and for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$, but $x \notin P(R')$. By feasibility, we have $\sum_{i \in N} p(R'_i) \geq \Omega$. Thus $x \notin P(R')$ implies that there exists $i \in N$ such that $x_i \geq p(R'_i)$. If $p(R'_i) \geq p(R_i)$, then $x_i \geq p(R_i)$, i.e. $x \notin P(R)$, a contradiction. Else, i.e., $p(R'_i) \leq p(R_i)$, we have, for all $M \subseteq N$, with $\mid M \mid \geq k+1$, for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$. If we take $M \equiv N$, then we must have $x_i \geq p(R_i)$, i.e. $x \notin P(R)$, a contradiction. Q.E.D.

The strict k-monotonicity condition implies k-monotonicity. According to Thomson (1990), the Pareto correspondence satisfies the no veto power condition. Therefore, the Pareto correspondence is $k-FTNE$-implementable

[3]See Sprumont (1991).

by Eliaz's result. Thus, our result coincides with that of Eliaz (2002). However, the following solutions of the no-envy correspondence, the individually rational correspondence from equal division, the core correspondence from equal division, the group no-envy correspondence and also all their intersections do not satisfy the no veto power condition. Thus, Eliaz's result (2002) can not inform us about the $k-FTNE$-implementability of these correspondences.

b) No-Envy correspondence, NE, (Foley, 1967)

The no-envy correspondence selects the allocations at which no agent prefers someone else´s consumption to his own. Formally:
Let $R \in \mathscr{D}$, $NE(R) = \{x \in X \text{ if } x_i R_i x_j \text{ for all } i, j \in N\}$.

Proposition 3 *: Let $0 \leq k < n$. The No-Envy correspondence satisfies strict k-monotonicity.*

Proof. Let $R, R' \in \mathscr{D}$, $x \in X$ and $x \in NE(R)$. Suppose that $x_i \leq p(R_i)$ (similar statements can be proved for $x_i \geq p(R_i)$). Suppose that no-envy correspondence does not satisfies strict k-monotonicity. Therefore, for all $M \subseteq N$, with $\mid M \mid \geq k+1$ and for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$, but $x \notin NE(R')$. Therefore, there are i, $j \in N$ such that $x_j P'_j x_i$, i.e., $x_j \notin L(x_i, R'_i)$. We have for all $M \subseteq N$, with $\mid M \mid \geq k+1$ and for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$. By taking $M \equiv N$, $x_j \notin LS(x_i, R_i) \cup \{x_i\}$. There are two cases:

i) The number $r_i(x_i)$ exists. In this case we have either $x_j \in]x_i, r_i(x_i)[$, i.e., $x_j P_j x_i$, and hence $x \notin NE(R)$, a contradiction. Or $x_j \in [r_i(x_i), \Omega]$, i.e. $r_j(x_j) \in LS(x_i, R_i) \cup \{x_i\}$. But $r_j(x_j) \sim_j x_j$, i.e., $x_j \in LS(x_i, R_i) \cup \{x_i\}$ which contradicts our assumption that $x_j \notin LS(x_i, R_i) \cup \{x_i\}$.

ii) The number $r_i(x_i)$ does not exist. In this case $x_j P_j x_i$ and hence $x \notin NE(R)$, a contradiction. Q.E.D.

c) Individually Rational Correspondence from Equal Division, I_{ed}

$I_{ed}(R) = \{x \in X : x_i R_i (M/n) \text{ for all } i \in N\}$

Proposition 4 *: Let $0 \leq k < n$. The Individually Rational Correspondence from Equal Division satisfies strict k-monotonicity.*

Proof. Omitted, it is very similar that of proposition 3.

d) Core Correspondence from Equal Division, C_{ed}

$C_{ed}(R) = \{x \in X : \nexists S \subseteq N$ and $(x'_i)_{i \in S}$ such that $(i) \sum_{i \in S} x'_i = \mid S \mid \Omega/n$, (ii) for all $i \in S$, $x'_i R_i x_i$ and for some $i \in S$, $x'_i P_i x_i\}$

Proposition 5 *: Let $0 \leq k < n$. The Core Correspondence from Equal Division satisfies strict k-monotonicity.*

Proof. Let $R, R' \in \mathscr{D}$ and $x, x' \in X$. Let $x \in C_{ed}(R)$. Suppose that $\sum_{i \in N} p(R_i) \geq \Omega$ (similar statements can be proved for $\sum_{i \in N} p(R_i) \leq \Omega$). Suppose that Core Correspondence from Equal Division does not satisfies strict k-monotonicity. Therefore, for all $M \subseteq N$, with $| M | \geq k+1$ and for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$, but $x \notin C_{ed}(R')$. We have $C_{ed}(R') \subseteq P(R')$. Therefore, we have two cases:

(i) $x \notin C_{ed}(R')$ and $x \notin P(R')$. In this case, we use the same arguments used in the proof of proposition 2.

(ii) $x \notin C_{ed}(R')$ and $x \in P(R')$. In this case, for all $i \in N$, $x_i \leq p(R'_i)$. There are two subcases:

α) $p(R_i) \leq p(R'_i)$. (a_1) If $x_i \geq p(R_i)$, then $x \notin P(R)$, therefore $x \notin C_{ed}(R)$, a contradiction. (a_2) If $x_i \leq p(R_i)$, since $x \notin C_{ed}(R')$, therefore $\exists S \subseteq N$ and $(x'_i)_{i \in S}$ such that (1) $\sum_{i \in S} x'_i = | S | \Omega/n$, (2) for all $i \in S$, $x'_i R'_i x_i$ and for some $i \in S$, $x'_i P'_i x_i$, then $(x_i)_{i \in S} < (x'_i)_{i \in S}$. Since we have $x_i \leq p(R_i)$ for $i \in N$, and we have for all $M \subseteq N$, with $| M | \geq k+1$ and for some $i \in M$, $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_j, R'_j)$ for all $j \in M$, by taking $M \equiv N$, there is $S \subseteq N$ and $(x'_i)_{i \in S}$ such that (3) $\sum_{i \in S} x'_i = | S | \Omega/n$, (4) for all $i \in S$, $x'_i R_i x_i$ and for some $i \in S$, $x'_i P_i x_i$. Thus, $x \notin C_{ed}(R)$, a contradiction.

β) $p(R'_i) \leq p(R_i)$. By feasibility, we have $\sum_{i \in N} p(R'_i) \geq \Omega$. We have also by statement (ii), $x \in P(R')$. Therefore for all $i \in N$, $x_i \leq p(R'_i)$. By the same arguments used in (a_2), we find that $x \notin C_{ed}(R)$, a contradiction. Q.E.D.

e) Group No-Envy correspondence, GNE

Let $R \in \mathscr{D}$, $GNE(R) = \{x \in X$: $\nexists S, S' \subseteq N$ with $| S | = | S' |$ and $(y_i)_{i \in S}$ such that $\sum_S y_i = \sum_{S'} x_i$, for all $i \in S$, $y_i R_i x_i$, and for some $i \in S$, $y_i P_i x_i\}$.

Proposition 6 *: Let $0 \leq k < n$. The Group No-Envy correspondence satisfies strict k-monotonicity.*

Proof. Since it is very similar that of proposition 5, we have omitted it.

Corollary 1 *: The Pareto correspondence, the no-envy correspondence, the individually rational correspondence from equal division, the core correspondence from equal division and the group no-envy correspondence are all implementable in k-FTNE by theorems 1 and 3.*

4.1 Stability and $k - FTNE$ implementability of the intersection of fair division solutions

In this subsection, we introduce the notion of stability by intersection. We notice that our conditions are preserved under intersection. This tool of stability allow us to identify directly the $k - FTNE$ implementability of various correspondences.

Definition 12 *(Stability by intersection, Doghmi and Ziad (2008 a))*
A given condition (C) is stable by intersection, if for any SCCs F and G satisfying (C) then the correspondence defined by $F \cap G \neq \emptyset$ satisfies (C)

Remark 1 *The k-strict-monotonicity, the strict weak k-no veto power, unanimity and condition (ii) of theorem 3 are stable by intersection.*

Corollary 2 *: The $(P \cap NE)$ correspondence, the $(P \cap I_{ed})$ correspondence, and the $(NE \cap I_{ed})$ correspondence are all implementable in k-FTNE by theorems 1 and 3.*

4.2 On no-monotonic solutions of the problem of fair division

In this subsection, we study the possible $k - FTNE$ implementability of no-monotonic solutions. We begin by the following solution: it select the allocation for which each agent holds a share proportionally with its preferred consumption if at least one preferred consumption is positive, and an average share if the preferred consumption of each agent equals zero.

f) Proportional solution, Pro

Let $R \in \mathscr{D}$, $x = Pro(R)$ if $x \in X$ and (i) when $\sum_{i \in N} p(R_i) \geq 0$, and $\exists \lambda \in \mathbb{R}_+$ s.t. $\forall i \in N$, $x_i = \lambda p(R_i)$; and (ii) when $\sum_{i \in N} p(R_i) = 0$, $x = (\Omega/n, ..., \Omega/n)$.

Proposition 7 *: The Proportional solution does not satisfies weak k-monotonicity.*

Proof. The Proportional solution is a *function*, i.e. the choice set is a singleton. In this case, k-monotonicity and weak k-monotonicity are equivalent. We have k-monotonicity implies Maskin monotonicity. The Proportional solution does not satisfy monotonicity. Therefore, it does not satisfies weak k-monotonicity. Q.E.D.

The following solution is quite simply a version of the proportional solution where the units of the good are treated symmetrically above or below the preferred consumptions.

g) Symmetrically proportional solution, SPro

Let $R \in \mathscr{D}$, $x = SPro(R)$ if $x \in X$ and (i) when $\sum_{i\in N} p(R_i) \geq \Omega$, and $\exists\lambda \in \mathbb{R}_+$ s.t. $\forall i \in N$, $x_i = \lambda p(R_i)$; and (ii) when $\sum_{i\in N} p(R_i) \leq \Omega$, $\exists\lambda \in \mathbb{R}_+$ s.t. $\Omega - x_i = \lambda(\Omega - p(R_i))$ for all $i \in N$.

Proposition 8 *: The Symmetrically proportional solution does not satisfies weak k-monotonicity.*

Proof. We omit the proof of this proposition. It is very similar of that of proposition 7.

Contrary to the proportionality, the solution below compares distances from preferred consumptions unit for unit. It selects the allocation at which all agents have same distance from their preferred consumptions, except when boundary problems occur; in this case, the agents which would have negative consumptions, instead get zero.

h) Equal-distance correspondence, Dis

Let $R \in \mathscr{D}$, $x = Dis(R)$ if $x \in X$ and (i) when $\sum_{i\in N} p(R_i) \geq \Omega$, $\exists d \geq 0$ s.t. $\forall i \in N$, $x_i = max\{0, p(R_i) - d\}$; and (ii) when $\sum_{i\in N} p(R_i) \leq \Omega$, $\exists d \geq 0$ s.t. $x_i = p(R_i) + d$ for all $i \in N$.

Proposition 9 *: The Equal-distance correspondence does not satisfy weak k-monotonicity.*

Proof. We omit the proof of this proposition. It is very similar to that of proposition 7.

i) Equal-sacrifice correspondence, Sac

This solution is based on the idea of the measurement of "the sacrifice" at allocation x by the size of agent i's upper contour set at x_i. It selects efficient allocations at which sacrifices are equal across agent, except when boundary problems occur; in this case, the agents which would have negative consumptions, instead get zero.

Let $R \in \mathscr{D}$, $x \in Sac(R)$ if $x \in X$ and (i) when $p(R) \geq M$, $\exists \sigma \geq 0$ s.t. $\forall i \in N$, $r_i(x_i) - x_i \leq \sigma$, strict inequality holding only if $x_i = 0$; and (ii) when $\sum_{i \in N} p(R_i) \leq \Omega$, $\exists \sigma \geq 0$ s.t. $x_i - r_i(x_i) = \sigma$ for all $i \in N$.

Proposition 10 *: The Equal-sacrifice correspondence does not satisfy weak k-monotonicity.*

Proof. We omit the proof of this proposition. It is very similar to that of proposition 7.

By propositions 7-10 and Eliaz's result (2002) of necessity, we have the following corollary:

Corollary 3 *The Proportional correspondence, the Symmetrically proportional correspondence, the Equal-distance correspondence and the Equal-sacrifice correspondence are all not implementable in k-FTNE.*

5 Application to finite allocation problems

5.1 $k - FTNE$ implementability of the Pareto correspondence

The weak Pareto correspondence is the solution which associates each economy with its feasible allocation set such that some agent weakly prefers to at any other feasible allocation. Formally,

$P_w(R) = \{x \in X : \exists i \in N \text{ such that } xR_iy \text{ for all } y \in X\}$.

Eliaz (2002) showed that the weak Pareto correspondence satisfies the necessary condition of $k - FTNE$ implementability, i.e., *weak k-monotonicity.* Also, it satisfies the no veto power condition. For sufficiency, we show that the weak Pareto correspondence satisfies k-monotonicity. [4]

Proposition 11 *: The weak Pareto correspondence satisfies k-monotonicity.*

Proof. Let $R, R' \in \Re$ and $x, y \in X$. Let $x \in P_w(R)$. Suppose that weak Pareto correspondence does not satisfy k- monotonicity. Therefore, for all

[4] The weak Pareto correspondence is a monotonic social choice correspondence that satisfies no veto power in any environment, and hence it can be implemented in Nash equilibria.

$M \subseteq N$, with $| M | \geq k+1$ and for some $i \in M$, $L(x, R_i) \subseteq L(x, R'_j)$ for all $j \in M$ (*), but $x \notin P_w(R')$. Therefore, there does not exist $i \in N$ such that for all $y \in X$, $xR'_i y$, i.e., there does not exist $i \in N$ such that $L(x, R'_i) = X$. By taking $M \equiv N$, we must have by (*) that does there not exist $i \in N$ such that $L(x, R_i) = X$, i.e., $x \notin P_w(R)$, a contradiction. Q.E.D.

By proposition 1 of Eliaz (2002), we have the following corollary,

Corollary 4 *: The weak Pareto correspondence can be implemented in k-FTNE.*

Now, we return to the concept of Pareto correspondence studied in section 4. We revise the name of Pareto correspondence to mean (strong) Pareto correspondence. Formally, it is defined as follows,

$P_s(R) = \{x \in X : \nexists y \in X$ such that for all $i \in N$, $yR_i x$, and for some $i \in N$, $yP_i x\}$.

In this environment, differently from that of exchange economies with single-peaked studied in section 4, the strong Pareto correspondence is not monotonic and hence it is not Nash implementable. But, the no-existence of the logical relationship between Maskin monotonicity and weak k-monotonicity as suggested by Elaiz (2002), makes us curios to see whether there is a possible $k - FTNE$ implementability. However, the next proposition shows the opposite. The strong Pareto correspondence is not weakly k-monotonic and hence it is not $k - FTNE$ implementable.

Proposition 12 *: The strong Pareto correspondence does not satisfy weak k-monotonicity.*

Proof. Consider the following example. Let $N = \{1, 2, 3\}$, $k = 1$ and $X = \{x, y, z, w\}$. Let $R, R' \in \Re$ be defined by:

R:

R_1	R_2	R_3
xy	yz	xz
z	x	y
w	w	w

$P_s(R) = \{x, y, z\}$

R':

R'_1	R'_2	R'_3
xyz	yz	xyz
w	xw	w

$P_s(R') = \{y, z\}$

Let $M \subseteq N$ with $| M | \geq 2$. We have $\{x, y, z\} \in P_s(R)$. For $M = \{1, 2\}$, $L(\{x, y, z\}, R_{i=2}) \subseteq L(\{x, y, z\}, R'_{j=1,2})$. We preceed by the same reasoning for $M = \{1, 3\}$, $M = \{2, 3\}$, and $M = \{1, 2, 3\}$. We obtain that $\forall M \subseteq N$ with $| M | \geq 2$, and for some $i \in M$, $L(\{x, y, z\}, R_i) \subseteq L(\{x, y, z\}, R'_j)$ for all $j \in M$, but $x \notin P_s(R')$. Q.E.D.

Corollary 5 : *The strong Pareto correspondence is not implementable in k-FTNE.*

5.2 $k-FTNE$ implementability of the core correspondence in coalitional games

A coalitional game contains a finite set $N = \{1, ..., n\}$ of agents with $n \geq 3$, a nonempty set X of outcomes, a preference profile $R \in \Re$, and a characteristic function $\nu : 2^N \setminus \emptyset \rightarrow 2^X$. A coalition, denoted S, is simply a set of agents, i.e. a subset of N such that $S \subset N$. An outcome $x \in X$ is strongly blocked by a coalition S if there is a $y \in \nu(S)$ such that yP_ix for all $i \in S$. An outcome $x \in X$ is weakly blocked by a coalition S if there is a $y \in \nu(S)$ such that yR_ix for all $i \in S$ and yP_ix for some $i \in S$.

5.2.1 The weak core correspondence

The nonempty weak core of a coalitional game is the set of all outcomes that are not strongly blocked by any coalition S. Formally, the nonempty weak core of a coalitional game environment is represented by:

$$C_w(R) = \{x \in \nu(N) : \nexists S \subseteq N \text{ and } \nexists y \in \nu(S) \text{ such that } yP_ix, \quad \forall i \in S\}.$$

We can rewrite this definition as following.

$$C_w(R) = \{x \in \nu(N) : \forall S \subseteq N, \exists i \in S \text{ such that } xR_iy \ \forall y \in \nu(S)\}.$$

Proposition 13 : *The nonempty weak core correspondence satisfies k-monotonicity.*

Proof. Let $R, R' \in \Re$, $x, y \in X$ and $x \in C_w(R)$. Let for all $M \subseteq N$, with $| M | \geq k + 1$, and for some $i \in M$, $L(x, R_i) \subseteq L(x, R'_j)$ for all $j \in M$ (**), but $x \notin C_w(R')$. Thus, $\forall S \subset N$, $\nexists i \in S$ such that $xR'_iy \ \forall y \in \nu(S)$. Thus, $\forall S \subset N$ and $\forall y \in \nu(S)\}$, $\nexists i \in S$ such that $y \in L(x, R'_i)$. By taking $S \equiv M$, we must have by (**) that $\forall S \subset N$, $\nexists i \in S$ such that $xR_iy \ \forall y \in \nu(S)\}$, i.e., $x \notin C_w(R)$, a contradiction. Q.E.D.

Proposition 14 : *The nonempty weak core correspondence satisfies k-weak no veto power.*

Proof. Let $R, R' \in \Re$, $x, y \in X$ and $x \in C_w(R)$. Suppose that C_w does not satisfy weak k-no veto power. Therefore, for some $M \subseteq N$ with $| M | \leq k + 1$, and for some $i \in M$, $y \in L(x, R_i) \subseteq L(y, R'_j)$ for all $j \in M$ and

$L(y, R'_j) = X$ for all $j \in N \backslash M$, but $y \notin C_w(R')$. Therefore, $\forall S \subset N$, $\nexists i \in S$ such that $yR'_i z$ $\forall z \in \nu(S)$. Thus, $\forall S \subset N$, $\forall z \in \nu(S)$, $\nexists i \in S$ such that $z \in L(y, R'_i)$. We have for some $M \subseteq N$ with $\mid M \mid \leq k+1$, (i) for some $i \in M$, $L(x, R_i) \subseteq L(y, R'_j)$ for all $j \in M$, and (ii) $L(x, R_j) \subseteq X = L(y, R'_j)$ for all $j \in N \backslash M$. Therefore, $\forall S \subset N$, $\forall z \in \nu(S)$, $\nexists i \in S$ such that $z \in L(x, R_i)$, a contradiction, because $x \in C_w(R)$. Q.E.D.

Corollary 6 *: The nonempty weak core correspondence is implementable in k-FTNE by theorem 2.*

5.2.2 The strong core correspondence

The strong core of a coalitional game is the set of all outcomes that are not weakly blocked by any coalition S. Formally, the strong core of a coalitional game environment is represented by:

$C_s(R) = \{x \in \nu(N) : \nexists S \subseteq N$ and $\nexists y \in \nu(S)$ such that $yR_i x$ for all $i \in S$ and $yP_i x$ for some $i \in S$.

Since there is a logical relationship among strict k-monotonicity, k-monotonicity, and Maskin monotonicity, the no-monotonic strong core correspondence salsifies neither k-monotonicity nor strict k-monotonicity. Thus, the strong core correspondence satisfies neither our sufficient conditions nor that of Eliaz (2002). To know its possible $k - FTNE$ implementability, we have checked if it satisfies the necessary condition of weak k-monotonicity. The response is negative as the following proposition shows.

Proposition 15 *: The strong core correspondence does not satisfy weak k-monotonicity.*

Proof. Consider the following example. Let $N = \{1, 2, 3\}$, $k = 1$, and $X = \{x, y, z, w\}$. We define $\nu : 2^N \setminus \emptyset \rightarrow 2^X$ as $\nu\{1, 2\} = \{x, z\}$, $\nu\{1, 3\} = \{x, y\}$, $\nu\{2, 3\} = \{y, w\}$, $\nu\{N\} = \{X\}$, and $\nu\{S\} = \emptyset$ for all other coalitions S. Let $R, R' \in \Re$ be defined by:

R:

R_1	R_2	R_3
y	xw	z
xw	z	xy
z	y	w

$C_s(R) = \{y, z\}$

R':

R'_1	R'_2	R'_3
xyzw	w	yzw
	yz	x
	x	

$C_s(R') = \{w\}$

In this example, we have $\{y, z\} \in C_w(R)$ and for all $M \subseteq N$ with $\mid M \mid \geq 2$, (i) for some $i \in M$, $L(y, R_i) \subseteq L(y, R'_j)$ for all $j \in M$, and (ii) for some

$i \in M$, $L(z, R_i) \subseteq L(z, R'_j)$ for all $j \in M$ but $y \notin C_s(R')$. Therefore, C_s does not satisfy weak k-monotonicity. Q.E.D.

Corollary 7 *: The strong core correspondence is not implementable in k-FTNE.*

6 Application to Equilibrium theory: the constrained Walrasian correspondence with slack

In a special case of private ownership economy, i.e., *pure exchange economies*, Eliaz (1999, 2002) showed the $k-FTNE$ implementability of the constrained Walrasian *correspondence* and the constrained Walrasian *function*. In this setting, the no veto power condition is trivially checked. However, for the general case with production and with possibly satiated preferences, Sjöström (1990) showed that a general version of the constrained Walrasian correspondence, called *constrained Walrasian correspondence with slack* (Mas-Colell,1988,1992), does not satisfy no veto power [5]. Thus, Elaiz's results cannot inform us about the $k - FTNE$ implementability of this correspondence. In the following, we define the constrained Walrasian correspondence with slack in an environment of exchange economies with possibly satiated preferences, and we apply our results.

Let E be an economy formed by: (a) a finite number K of goods; (b) a finite number J of firms such that each firm j is specified by its nonempty production set $Z_j \subset \mathbb{R}^K$; (c) a finite number I of consumers such that each consumer i is specified by its nonempty consumption set $X_i \subset \mathbb{R}^K$. Let $B = \{p \in \mathbb{R}^K :\| p \|\leq 1\}$ be the price set. Every consumer i is endowed with a preference relation R_i which is irreflexive and transitive, an initial endowment of goods $w_i \in X_i$, and a share s_{ij} of the profits of firm j. The shares are non-negatives and they satisfy $\sum_{i=1}^{I} s_{ij} = 1$ for all j. We assume the nonempty consumption sets X_i and Z_j are closed and convex; and no free disposal is allowed.

Definition 13 *Let $p^* \in \mathbb{R}^K$ be a price vector, let $(z_j^*)_{j=1,\ldots,J}$ be a production plan and let $(x_i^*)_{i=1,\ldots,I}$ be a consumption plan. The triplet $(p^*, (z_j^*), (x_i^*))$ is a constrained Walrasian equilibrium with slack of the economy E, denoted $CE_s(E)$, if there is $\alpha \geq 0$ such that:*

[5] The constrained Walrasian correspondence with slack is Nash implementable (Sjöström (1990)).

i) $p^*.z_j^* \geq p^*.z_j$ *for all* $z_j \in Z_j$ *for all* j *(profit maximization);*
ii) $x_i^* R_i x_i$ *for all* $x_i \leq \sum_{i \in I} w_i$ *such that* $p^*.x_i \leq p^*.w_i + \sum_{j=1}^{J} s_{ij} p^*.z_j^* + \alpha$ *(preference maximization);*
iii) $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} z_j^*$ *(feasibility).*

Remark 2 *If* $\alpha = 0$ *the constrained Walrasian equilibrium with slack becomes quite simply the usual definition of constrained Walrasian equilibrium in the general case with production. If* $\alpha = 0$ *and the available production technology is given by* $Z = \mathbb{R}_-^K$*, the constrained Walrasian equilibrium with slack reduce to constrained Walrasian equilibrium introduced by Hurwicz, Maskin and Postlewaite (1995).*

Proposition 16 *: The constrained Walrasian correspondence with slack satisfies* k*-monotonicity.*

Proof. Let $R, R' \in \Re$, $x^* \in X_I$, $z^* \in Z_J$ and $(p^*, x^*, z^*) \in CE_s(E)$ where $p^* \in \mathbb{R}_+^K$. Suppose that CE_s does not satisfy k-monotonicity. Therefore, for all $M \subseteq I$, with $\mid M \mid \geq k+1$, and for some $i \in M$, $L(x_i^*, R_i) \subseteq L(x_j^*, R_j')$ for all $j \in M$, but $x^* \notin CE_s(E')$. Thus, $\exists j \in I$ and $\exists y \in X_I$ such that $y_j P_j' x_j^*$ and $p^*.y_i \leq p^*.w_i + \sum_{j=1}^{J} s_{ij} p^*.z_j^* + \alpha$ for all $i \in I$. Thus, $y_j \notin L(x_j^*, R_j')$. By taking $M \equiv I$, $y_j \notin L(x_j^*, R_j)$, i.e., $y_j P_j x_j^*$, a contradiction, because $x^* \in CE_s(E)$. Q.E.D.

Proposition 17 *: The constrained Walrasian correspondence with slack satisfies weak* k*-no veto power.*

Proof. Let $R, R' \in \Re$, $x^* \in X_I$, $z^* \in Z_J$ and $(p^*, x^*, z^*) \in CE_s(E)$ where $p^* \in \mathbb{R}_+^K$. Suppose that CE_s does not satisfies weak k-no veto power. Therefore, for some $M \subseteq I$ with $\mid M \mid \leq k+1$, for some $i \in M$, $x_i' \in L(x_i^*, R_i) \subseteq L(x_j', R_j')$ for all $j \in M$, and $L(x_j', R_j') = X_I$ for all $j \in I \backslash M$, but $x' \notin CE_s(E')$. Thus, $\exists j \in I$ and $\exists y \in X_I$ such that $y_j P_j' x_j'$ and $p^*.y_i \leq p^*.w_i + \sum_{j=1}^{J} s_{ij} p^*.z_j^* + \alpha$ for all $i \in I$. Thus, $y_j \notin L(x_j', R_j')$. We have for some $i \in M$, $L(x_i^*, R_i) \subseteq L(x_j', R_j')$ for all $j \in M$ and $L(x_j^*, R_j) \subseteq X_I = L(x_j', R_j')$ for all $j \in I \backslash M$. Therefore, $y_j \notin L(x_j^*, R_j)$, i.e., $y_j P_j x_j^*$, a contradiction, because $x^* \in CE_s(E)$. Q.E.D.

Corollary 8 *: The constrained Walrasian correspondence with slack is implementable in* k*-FTNE by theorem 2.*

7 Conclusion

We have extended theorems 1 and 2 of Doghmi and Ziad (2008 *a*) to bounded rationality environments by reexamining Eliaz's results (2002) of fault tolerant implementation. We have identified two combinations of sufficient conditions to implement SCCs in $k - FTNE$: (1) strict k-monotonicity, strict k-no veto power, and unanimity, (2) k-monotonicity, k-no veto power, and unanimity. In addition, these conditions are stable by intersection which makes it possible to check in a direct way the $k - FTNE$ implementability of various social choice correspondences. We have applied these results to exchange economies with single -peaked preferences, to finite allocation problems, and to equilibrium theory. Firstly, we have noted, contrary to Eliaz's results (2002), that our conditions are satisfied by all monotonic solutions and particularly those which do not satisfy the no-veto power condition. Secondly, in exchange economy when preferences are single-peaked, our condition of k-strict weak no-veto power always holds independently of the solution. Thus, in these environments, the k-monotonicity is, alone, sufficient for the $k - FTNE$ implementation for the *correspondences* and both necessary and sufficient for *functions*. The no-existence of the logical relationship between Maskin monotonicity and weak k-monotonicity as suggested by Elaiz (2002), was encouraging for a possible $k - FTNE$ implementability. Not all was well however. Thus, to find an application to example 4 (p 603)of Eliaz (2002) remains an open question. We can also propose two others questions for future research.

Firstly, we can study the degree of faultness. If, for example, the planner imposes a cost for the players who reach certain degree, what occurs in this case?

Secondly, if in exchange economies with single peaked preferences, the agents have initial endowments, are the correspondences studied here Nash implementable and/or $k - FTNE$ implementable ?

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