

# Parimutuel Betting under Asymmetric Information\*

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## Abstract

This paper examines simple parimutuel betting games under asymmetric information, with particular attention to differences between markets in which bets are submitted simultaneously versus sequentially. In the simultaneous parimutuel betting market, all (symmetric and asymmetric) Bayesian-Nash equilibria are generically characterized as a function of the number of bettors and the quality of their private information. There always exists a separating equilibrium, in which all bettors follow their private signals. This equilibrium is unique if the number of bettors is sufficiently large. In the sequential framework, earlier bets have information externalities, because they may reveal private information of bettors. They also have payoff externalities, because they affect the betting odds. One effect of these externalities is that the separating equilibrium disappears if the number of betting periods is sufficiently large.

KEYWORDS: Parimutuel betting; Asymmetric information; Information aggregation; Herd behavior; Contrarian behavior.

JEL CLASSIFICATION: C72; D82.

## 1 Introduction

In 2002, Deutsche Bank and Goldman Sachs introduced options on economic data releases such as employment, retail sales, industrial production, inflation, and economic growth, with the purpose of providing a means of hedging core risk. These new economic derivatives are priced and allocated *parimutuelly*, meaning that their prices are based solely on the relative

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demand for their implied outcomes. The parimutuel system is also the accepted betting procedure at major horse-racing tracks throughout the world, where investments on the winning horse yield returns that are decreasing with the proportion of bettors who have bet on the same horse.

Several empirical and theoretical studies have investigated the properties of parimutuel betting markets and have pointed out their relevance for the analysis of decision making under risk and market efficiency.<sup>1</sup> Empirical research establishes that, although there is a tendency in parimutuel betting markets for price changes to move in the direction of actual outcomes over time, some empirical regularities are clearly inconsistent with informational efficiency. The most robust anomalous empirical regularity, called the *favorite-longshot bias*, is characterized by the win pool shares being lower than win frequencies for favorites, and higher than win frequencies for longshots. It is thus generally more profitable to bet on favorites than on longshots.

Most of the existing theoretical studies of parimutuel betting have focused on games with symmetric information, either by modeling bettors as having homogeneous beliefs (Chadha and Quandt, 1996; Feeney and King, 2001), including uninformed bettors modeled as noise bettors (Hurley and McDonough, 1995; Terrell and Farmer, 1996; Koessler, Ziegelmeier, and Broihanne, 2003), or endowing all bettors with inconsistent beliefs (Watanabe, Nonoyama, and Mori, 1994; Watanabe, 1997). In contrast to previous studies, the model of parimutuel betting introduced here considers an environment in which differences in beliefs are due only to differences in privately-held information about the probability of each outcome. For the example of a horse race, such differences in information may be due, e.g., to the dispersion of knowledge concerning the intrinsic ability of each horse, the condition of the track, the skill of each jockey, the horses' performances in previous races, etc. . . . Our focus is on the properties of equilibria of betting games in this environment and their relationship to the underlying informational structure.

The most closely related theoretical analysis of parimutuel betting is a recent model of Ottaviani and Sørensen (2005). They propose two models of parimutuel betting with, as in our model, risk-neutral players, two states of nature, and a prior probability distribution over states that is common for all bettors. In the first of their two models, there is a finite number of symmetrically informed bettors who choose the time and the exact amount of money they want to bet on an outcome. In their second model, there is a continuum of privately informed bettors who bet an indivisible unit of money simultaneously. Ottaviani and Sørensen (2005) show, among other results, that both types of models can generate a favorite-longshot bias.

In this paper, contrary to Ottaviani and Sørensen (2005), we model the timing of bets as exogenous and analyze the difference between simultaneous and sequential betting. We

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<sup>1</sup>A detailed account of this literature may be found in Sauer (1998) and Vaughan Williams (1999).

first consider the simultaneous move version of the game. Generically (for almost all bettors' qualities of information), we characterize the set of all (symmetric and asymmetric) pure strategy Bayesian-Nash equilibria as a function of the number of bettors and the quality of their private information. In particular, we show that a separating equilibrium, in which all bettors always bet consistently with their private information, always exists. Asymmetric equilibria, in which some players bet on the same horse whatever their private signal, also exist. Although such a multiplicity of equilibria raises selection questions, we show that the selection issue is irrelevant with a large number of bettors, since asymmetric equilibria vanish when the number of bettors increases. Consequently, in large, simultaneous wagering markets, the odds against each horse always reflect all private information. However, as in Ottaviani and Sørensen's (2005) second model, a favorite-longshot bias is obtained because market odds, which reflect individuals' average beliefs, are less extreme than the beliefs that would have been obtained from the aggregation of all private signals.

Next, we investigate a market in which bets are submitted sequentially. The sequential version takes into account information externalities, resulting from the inferences later bettors can draw from the decisions of earlier bettors, and the related strategic incentives to influence later bettors. It also includes payoff externalities resulting from earlier bettors' influence on the odds, and thus the payoffs to each action, that later bettors face. We show that a separating equilibrium can fail to exist, and will typically disappear when the number of bettors increases. This non-existence is related to the fact that bettors exhibit *herd behavior*, as in standard models of information cascades (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992), which is the imitation of predecessors' choices while overriding own private information. However, as in most multi-agent sequential trade models with asymmetric information (e.g., Avery and Zemsky, 1998), the price mechanism ensures that complete imitation does not continue indefinitely.

The paper proceeds as follows: Section 2 lays out the basic model of parimutuel betting used throughout the paper. In Section 3, we present the results concerning the simultaneous move betting game. Section 4 investigates the sequential move version of the game. We relate our study to earlier work on social learning in multi-person sequential decision problems in Section 5. The appendix contains the proofs for some of the results of Sections 3 and 4.

## 2 A Model of Parimutuel Betting

We consider an environment with two possible outcomes, which for ease of exposition can be thought of as a horse race where there are only two horses. Each member of a group of

bettors is required to bet one unit of money on one of the two outcomes.<sup>2</sup> We consider a finite set of asymmetrically informed bettors. Each player is endowed with beliefs about the state of Nature obtained from his signal and from the information contained in the choices of his predecessors.

More precisely, the market is modeled as follows. The two horses are called  $A$  and  $B$ . There is a finite set  $N = \{1, \dots, n\}$  of risk neutral bettors. Each bettor  $i \in N$  independently chooses to bet one unit of money on a horse  $s_i \in S_i \equiv \{A, B\}$ . For any profile of bets  $s = (s_1, \dots, s_n) \in S = \prod_{i \in N} S_i$  and any horse  $H \in \{A, B\}$ , let  $H(s) = \{i \in N : s_i = H\}$  be the set of bettors who bet on horse  $H$ , let  $h(s) = |H(s)|$  be the number of bettors who bet on horse  $H$ , and let  $\bar{H}$  be the horse other than  $H$ . The *odds against horse  $H$* , which is given by the total number of bets on horse  $\bar{H}$  divided by the total number of bets on horse  $H$ , is denoted by

$$O_H(s) = \frac{n - h(s)}{h(s)}.$$

Bettors have a flat common prior belief about the set of states of Nature  $\{\theta_A, \theta_B\}$ , where  $\theta_A$  stands for “horse  $A$  wins” and  $\theta_B$  stands for “horse  $B$  wins”.

If bettor  $i$  bets on the winning horse, then his payoff is normalized to the return of this horse, which is equal to its odds plus 1. Otherwise, bettor  $i$  receives 0 payoff. For example, odds of 4 to 1 laid against a horse implies a payoff to a successful bet of four units of money, plus the stake returned. On the contrary, an unsuccessful bet loses the stake. Formally, each bettor  $i \in N$  has a (vNM) utility function  $u_i : S \times \{\theta_A, \theta_B\} \rightarrow \mathbb{R}$  such that for all  $H \in \{A, B\}$ ,

$$u_i(s, \theta) = \begin{cases} O_H(s) + 1 = \frac{n}{h(s)} & \text{if } s_i = H \text{ and } \theta = \theta_H \\ 0 & \text{if } s_i = H \text{ and } \theta \neq \theta_H. \end{cases} \quad (1)$$

Before taking his decision, and in addition to the possible history of choices he has observed, each bettor  $i \in N$  gets a *private signal*  $q_i \in Q_i \equiv \{q^A, q^B\}$  that is correlated with the true state of nature. Conditional on the state of nature, bettors’ signals are i.i.d. and satisfy

$$\begin{aligned} \Pr(q_i = q^A \mid \theta_A, q_j) &= \Pr(q_i = q^A \mid \theta_A) = \pi > 1/2 \\ \Pr(q_i = q^A \mid \theta_B, q_j) &= \Pr(q_i = q^A \mid \theta_B) = 1 - \pi, \end{aligned}$$

for all  $i, j \in N, i \neq j$ . Hence, once bettor  $i$  has received a signal  $q_i \in Q_i$ , his beliefs about the states of Nature are given by  $\Pr(\theta_A \mid q^A) = \Pr(\theta_B \mid q^B) = \pi$  and  $\Pr(\theta_A \mid q^B) = \Pr(\theta_B \mid q^A) = 1 - \pi$ . The parameter  $\pi$  characterizes bettors’ *quality of information*.

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<sup>2</sup>If rational bettors were able to drop out of betting, a no-trade result would apply. A complementary analysis to the one presented in this paper would be to investigate rational parimutuel betting with the possibility of withdrawal, assuming that there are some players who bet for exogenous reasons.

### 3 Simultaneous Betting

To avoid the complexities associated with social learning and signaling of sequential games, we first analyze the simultaneous parimutuel betting game. In such a setting, a bettor's strategy is a rule of action which maps each realization of his signal to one of two actions: to bet on horse  $A$ , or to bet on horse  $B$ . More precisely, a (pure) strategy of bettor  $i$  is a mapping  $\sigma_i : Q_i \rightarrow S_i$ . Bettor  $i$ 's belief that the winning horse is horse  $H$ , and that other bettors receive the profile of signals  $q_{-i}$ , is given by  $\Pr(\theta_H, q_{-i} | q_i) = \Pr(q_{-i} | \theta_H) \Pr(\theta_H | q_i)$ . Thus, bettor  $i$ 's expected utility given others' strategies  $\sigma_{-i}$  when he bets on horse  $H$  and receives the signal  $q_i$  can be written as

$$U_i(H, \sigma_{-i} | q_i) = \Pr(\theta_H | q_i) \sum_{q_{-i} \in Q_{-i}} \frac{\Pr(q_{-i} | \theta_H) n}{h(H, \sigma_{-i}(q_{-i}))}.$$

In a *Bayesian-Nash equilibrium*, the action prescribed by each bettor's strategy maximizes his expected payoff conditional on his signal given others' strategies.<sup>3</sup> The following lemma establishes that there is no equilibrium where any bettor bets against his private signal, regardless of the content of the signal.

**Lemma 1** *For any number of bettors,  $n$ , and any quality of information,  $\pi$ , any strategy profile in which bettor  $i$  bets unconditionally against his private signal, i.e.,  $\sigma_i(q^H) = \bar{H}$  for all  $H \in \{A, B\}$ , cannot constitute a Bayesian-Nash equilibrium.*

**Proof.** The strategy  $\sigma_i(q^H) = \bar{H}$  for all  $H$  is optimal for bettor  $i$  if  $U_i(\bar{H}, \sigma_{-i} | q^H) \geq U_i(H, \sigma_{-i} | q^H)$  for all  $H$ , which is impossible with  $\pi > 1/2$ . ■

On the contrary, a natural and intuitive strategy for a bettor is simply to follow his private signal, i.e., to bet on horse  $A$  if he receives the signal  $q^A$  and to bet on horse  $B$  if he receives the signal  $q^B$ . In the following lemma we show that there is always a *separating equilibrium* of this form, and this equilibrium is the unique symmetric equilibrium.

**Lemma 2** *For any number of bettors,  $n$ , and any quality of information,  $\pi$ , there exists a separating equilibrium, in which every bettor follows his private signal, i.e.,  $\sigma_i(q^H) = H$  for all  $H \in \{A, B\}$  and  $i \in N$ . This equilibrium is the unique symmetric equilibrium.*

**Proof.** Let  $i \in N$  and assume that  $\sigma_j(q^H) = H$  for all  $H \in \{A, B\}$  and  $j \neq i$ . Then, for any  $H \in \{A, B\}$ ,  $U_i(H, \sigma_{-i} | q^H) \geq U_i(\bar{H}, \sigma_{-i} | q^H)$  is satisfied if and only if  $\Pr(\theta_H | q^H) \geq \Pr(\theta_{\bar{H}} | q^H)$ , i.e.,  $\pi \geq 1 - \pi$ , which is satisfied since  $\pi > 1/2$ . To prove that the separating equilibrium is the unique symmetric equilibrium, notice that there are three other symmetric

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<sup>3</sup>To simplify the exposition, we ignore cases of indifference throughout the paper when the indifference is due to non-generic values of  $\pi$ .

strategy profiles: (i) every bettor bets against his own private signal, (ii) every bettor bets on horse  $A$ , and (iii) every bettor bets on horse  $B$ . None of those strategy profiles form a Bayesian-Nash equilibrium. The result that (i) is not an equilibrium follows directly from Lemma 1. To show that (ii) is not an equilibrium, let  $\sigma_i(q^H) = A$  for all  $H \in \{A, B\}$  and  $i \in N$ . Then,  $U_i(B, \sigma_{-i} | q^B) = \Pr(\theta_B | q^B) \frac{n}{1} = n\pi$  and  $U_i(A, \sigma_{-i} | q^B) = \Pr(\theta_A | q^B) \frac{n}{n} = (1 - \pi) < n\pi$ . Hence,  $\sigma$  is not an equilibrium. An analogous argument shows that (iii) is not an equilibrium. ■

We now investigate the possibility of asymmetric equilibria. An equilibrium with exactly  $k$  separating strategies is called a  $k$ -separating equilibrium. (Recall that for any non-negative integers  $j$  and  $k$ , the binomial coefficient is denoted by  $C_k^j \equiv \frac{k!}{j!(k-j)!}$ .)

**Lemma 3** *There exists a  $k$ -separating equilibrium, where exactly  $k \in \{0, 1, \dots, n-2\}$ bettors always follow their private signal, if and only if  $n - k$  is an even number,  $\frac{n-k}{2}$ bettors always bet on horse  $A$ ,  $\frac{n-k}{2}$ bettors always bet on horse  $B$ , and the following inequality is satisfied:*

$$\frac{\pi}{1 - \pi} \leq \frac{\sum_{j=0}^k \frac{C_k^j \pi^j (1 - \pi)^{k-j}}{\frac{n-k}{2} + j}}{\sum_{j=0}^k \frac{C_k^j \pi^j (1 - \pi)^{k-j}}{\frac{n-k}{2} + 1 + j}}. \quad (2)$$

*If such an equilibrium exists, then there are exactly  $C_n^k \times C_{\frac{n-k}{2}}^{\frac{n-k}{2}}$  equilibria with exactly  $k$  separating strategies.*

**Proof.** See the Appendix. ■

For example, if  $k = 0$  and  $n$  is an even number, then from the previous lemma there is an equilibrium, in which half of thebettors always bet on horse  $A$  and the other half always bet on horse  $B$ , if and only if  $\pi \leq \frac{n+2}{2n+2}$ . If  $k = 1$  and  $n$  is an odd number, then an equilibrium exists in which  $(n-1)/2$ bettors always bet on horse  $A$ ,  $(n-1)/2$ bettors always bet on horse  $B$ , and  $\pi \leq \frac{1}{1 + \sqrt{\frac{n-1}{n+3}}}$ .

We are not able to explicitly solve inequality (2) for  $\pi$  with arbitrarily values of  $k$ . However, we provide the following lemma, which reveals an interesting property of the solution. The lemma states that the polynomial in  $\pi$  induced by inequality (2) has one and only one real root on the interval  $(1/2, 1)$ . More precisely, as in the previous examples with  $k = 0$  and  $k = 1$ , partially separating equilibria as described in Lemma 3 exist only for sufficiently low qualities of information.

**Lemma 4** *For all  $n \geq 2$  and  $k < n$  such that  $n - k$  is an even number, there exists  $\pi(k, n) \in (1/2, 1)$  such that inequality (2) is satisfied if and only if  $\pi \leq \pi(k, n)$ .*

**Proof.** See the Appendix. ■

It can be numerically verified that  $\pi(k, n)$  is increasing in  $k$  and decreasing in  $n$ .<sup>4</sup> That is, conditions for the existence of a  $k$ -separating equilibrium are weaker as the number of separating strategies,  $k$ , increases, but become stronger as the total number of bettors increase. The next proposition shows that the separating equilibrium is in fact unique for sufficiently large parimutuel betting markets.

**Proposition 1** *For all  $\pi > 1/2$ , there exists  $\bar{n}$ , such that for all  $n \geq \bar{n}$ , the unique Bayesian-Nash equilibrium of the  $n$ -player simultaneous game is the separating equilibrium.*

**Proof.** See the Appendix. ■

Although betting odds perfectly reflect all private information in large parimutuel markets, the equilibrium distribution of bets always underestimates the winning probability of the favorite. The favorite is the outcome on which the majority of bettors have bet, while the long shot is the outcome on which a minority of bettors have bet. When the number of bettors increases, the market's estimate of the winning chances of the favorite tend towards the quality of an individual bettor's private information (the probability  $\pi$  that any bettor receives a correct signal, which is less than one but greater than one-half). In contrast, the objective winning probability of the favorite, which an observer could deduce from the observation of the final distribution of bets, tends towards one. This phenomenon, called the *favorite-longshot bias*, is a well known empirical regularity of parimutuel betting markets, and implies that a bettor who is able to bet after all other bettors can make abnormal earnings by simply betting on the horse with the largest frequency of bets (the favorite). This bias resulting from simultaneous betting is also investigated by Ottaviani and Sørensen (2005) in a setting with a continuum of bettors and continuous private signal spaces.

## 4 Sequential Betting

While there are equilibria of the simultaneous betting game, in which the odds reflect all players' private information, bettors cannot use this publicly revealed information when they make their decisions. In this section, we assume that bets are made sequentially and prior bettors' decisions are publicly observable. This latter framework is more appealing since in modern parimutuel betting markets high-speed electronic calculators, known as totalizators or tote boards, record and display up-to-the-minute betting patterns. In a similar vein, detailed pricing, volatility, and probability information is provided electronically in real time during the economic derivatives auctions launched by Deutsche Bank and Goldman Sachs. At the time an individual makes a decision, the decisions of previous bettors are known and have been integrated into the current market odds, and the market is effectively sequential.

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<sup>4</sup>This has been checked numerically up to  $n = 100$  bettors for all possible values of  $k$ .

## 4.1 The Dynamic Parimutuel Game

At each stage in the sequence, bettors can observe the bets cast by preceding bettors, but not the signals that those earlier bettors received. A history of  $k$  decisions is denoted by  $s^k \in S^k = \prod_{j=1}^k \{A, B\}$ . When bettor  $k + 1$  takes a decision, he observes a history  $s^k$ , i.e., he observes the decisions taken by bettors  $1, \dots, k$ . Let period  $k$  denote the time at which player  $k$  makes his bet. Let  $H(s^k) = \{i \in \{1, \dots, k\} : s_i = H\}$  be the set of bettors up to period  $k$  who chose horse  $H$ , and let  $h(s^k) = |H(s^k)|$  be the associated number of bets on horse  $H$ .

A (pure) strategy for bettor  $i$  is a function  $\sigma_i : S^{i-1} \times Q_i \rightarrow S_i$ . Hence,  $\sigma_i(s^{i-1}; q_i)$  is the choice of bettor  $i$  with signal  $q_i$  when he has observed the history  $s^{i-1}$ . A profile of strategies is denoted by  $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma = \prod_{i \in N} \Sigma_i$ . Denote respectively by  $q_{i+} = (q_j)_{j>i}$  and  $\sigma_{i+} = (\sigma_j)_{j>i}$  the signals and strategies of the successors of bettor  $i$ , and let  $Q_{i+} = \prod_{j>i} Q_j$  and  $\Sigma_{i+} = \prod_{j>i} \Sigma_j$ . Let  $s(\sigma_{i+} | s^i; q_{i+})$  be the final history reached according to the profile of strategies  $\sigma_{i+} \in \Sigma_{i+}$  given the history  $s^i \in S^i$  and the profile of signals  $q_{i+} \in Q_{i+}$ . Let  $s(\sigma | q) = s(\sigma | \emptyset; q)$  be the final history reached according to the profile of strategies  $\sigma$  given the profile of signals  $q$ , and let  $s(\sigma_{n+} | s^n; q_{n+}) = s^n$ .

Bettor  $i$ 's belief about the state of Nature  $\theta$  when he has received the signal  $q_i$  and has observed the history  $s^{i-1}$  is denoted by  $\mu_i(\theta | s^{i-1}; q_i)$ . The conditional independence of the signals gives  $\Pr(\theta_H, q_{i+} | s^{i-1}, q_i) = \Pr(q_{i+} | \theta_H) \mu_i(\theta_H | s^{i-1}; q_i)$ . Hence, bettor  $i$ 's expected utility is:

$$U_i(H, \sigma_{i+} | s^{i-1}; q_i) = \mu_i(\theta_H | s^{i-1}; q_i) \mathbb{E} \left( \frac{n}{h(s(\sigma_{i+} | s^{i-1}, H; q_{i+}))} \mid \theta_H \right). \quad (3)$$

A pair  $(\sigma, \mu)$  is a *perfect Bayesian equilibrium* if  $\mu$  is obtained from  $\sigma$  by Bayes' rule whenever possible, and if  $\sigma$  satisfies sequential rationality. Formally, a pair  $(\sigma, \mu)$  is *sequentially rational* if for all  $i \in N$ ,  $q_i \in Q_i$ , and  $s^{i-1} \in S^{i-1}$  we have  $U_i(\sigma_i(s^{i-1}; q_i), \sigma_{i+} | s^{i-1}; q_i) \geq U_i(s_i, \sigma_{i+} | s^{i-1}; q_i)$ , for all  $s_i \in S_i$ . For all  $j \in N$ , let  $J_j(s_j | s^{j-1}) \equiv \{q_j \in Q_j : \sigma_j(s^{j-1}; q_j) = s_j\}$  be the set of signals such that bettor  $j$  plays  $s_j$  after he has observed the history  $s^{j-1} \in S^{j-1}$ . If  $J_j(s_j | s^{j-1}) \neq \emptyset$  for all  $j < i$ , then bettor  $i$  can apply Bayes' rule:  $\mu_i(\theta | s^{i-1}; q_i) = \Pr(\theta | q_i, q_j \in J_j(s_j | s^{j-1}) \forall j < i)$ . Hence, a pair  $(\sigma, \mu)$  is *weakly consistent* if for all  $i \in N$ ,  $q_i \in Q_i$ , and  $s^{i-1} \in S^{i-1}$  we have

$$\mu_i(\theta | s^{i-1}, q_i) = \frac{\Pr(q_i | \theta) \prod_{j<i} \sum_{q_j \in J_j(s_j | s^{j-1})} \Pr(q_j | \theta)}{\sum_{\theta_H \in \{\theta_A, \theta_B\}} \Pr(q_i | \theta_H) \prod_{j<i} \sum_{q_j \in J_j(s_j | s^{j-1})} \Pr(q_j | \theta_H)}, \quad (4)$$

whenever  $J_j(s_j | s^{j-1}) \neq \emptyset$  for all  $j < i$ .

The next lemma is an extension of Lemma 1 to the dynamic game. Whatever the history of bets, no player has an incentive to use a pure strategy of betting against his private signal.



**Lemma 5** For any number of bettors,  $n$ , and any quality of information,  $\pi$ , any strategy profile in which bettor  $i$  bets unconditionally against his private signal for some history  $s^{i-1}$ , i.e.,  $\sigma_i(s^{i-1}; q^H) = \bar{H}$  for all  $H \in \{A, B\}$ , cannot constitute a perfect Bayesian equilibrium.

**Proof.** Assuming that  $U_i(B | s^{i-1}, q^A) \geq U_i(A | s^{i-1}, q^A)$  and  $U_i(A | s^{i-1}, q^B) \geq U_i(B | s^{i-1}, q^B)$  with  $\pi > 1/2$  yields a contradiction, as in the simultaneous game. ■

The next lemma characterizes the optimal strategy of the last bettor in the sequence.

**Lemma 6** Bettor  $n$  bets on horse  $H$  if and only if  $\mu_n(\theta_H | s^{n-1}; q_n) \geq \frac{h(s^{n-1})+1}{n+1}$ .

**Proof.** The proof follows directly from the expected utility given in equation (3). ■

In the next subsection we characterize the separating equilibria to the dynamic betting game. In contrast to the simultaneous setting, a separating equilibria does not always exist in a sequential betting market. Furthermore, if separating equilibria do exist, they disappear as the number of bettors increases.

## 4.2 Separating Equilibria

From Lemma 5, we know that an equilibrium strategy  $\sigma_i$  for player  $i$  is separating if and only if player  $i$  always follows his own private signal, whatever the content of his signal and for all possible histories he has observed. Betting against his private signal for all possible signals cannot be part of an equilibrium. More generally, we say that a profile of strategies up to player  $i - 1$ ,  $(\sigma_j)_{j < i}$ , is separating if  $\sigma_j(s^{j-1}; q^H) = H$  for all  $j < i$ ,  $s^{j-1} \in S^{j-1}$ , and  $H \in \{A, B\}$ , that is, that the first  $i - 1$  players have bet in accordance with their signals. The next lemma characterizes bettors' updated beliefs when previous bettors have used separating strategies.

**Lemma 7** If the profile of strategies  $(\sigma_j)_{j < i}$  is separating then

$$\mu_i(\theta_H | s^{i-1}; q^H) = \frac{\pi^{2h+2-i}}{\pi^{2h+2-i} + (1-\pi)^{2h+2-i}} \text{ and } \mu_i(\theta_H | s^{i-1}; q^{\bar{H}}) = \frac{\pi^{2h-i}}{\pi^{2h-i} + (1-\pi)^{2h-i}},$$

for all  $H \in \{A, B\}$  and  $s^{i-1} \in S^{i-1}$ , where  $h = h(s^{i-1})$ .

**Proof.** The result follows from the application of Bayes rule (equation (4)) with  $J_j(H | s^{j-1}) = \{q^H\}$ . ■

The next proposition shows that there is a range of parameter values for which no separating equilibrium exists. More precisely, it is shown that if the quality of bettors' private information fails to be in a certain range, then a separating equilibrium does not exist. This

simple result illustrates the role of payoff and information externalities. The payoff externality of a bet is the effect it has on the odds that future bettors face. The information externality of a bet is the effect it has on the probability assessment of future bettors. Consider the last bettor in the sequence, bettor  $n$ . If the quality of information is too low and the private signal of bettor  $n$  is in agreement with the majority of past bets, his assessment of the probability the favorite wins is not sufficiently high to offset the potential high payout of betting on the longshot, and thus it is optimal for bettor  $n$  to bet against the trend and his own private signal. Such an action is called *contrarian behavior*, as the agent ignores his private signal and bets against the current favorite. On the other hand, if the quality of information bettors hold is too high, then previous bets provide strong evidence that the favorite will win, and thus bettor  $n$  would override his signal if it favors the longshot, and bet on the favorite. This action is called *herd behavior*. In this latter case, the fact that previous bettors have bet on the same horse reveals relevant information about the probability the horse wins, overwhelming the bettor's private assessment and more than offsetting the negative effect of the low payout in the event the favorite wins.

**Proposition 2** *Assume that  $n > 2$ . If  $(\sigma, \mu)$  is a separating equilibrium then*

$$\frac{n^{1/(n-2)}}{1 + n^{1/(n-2)}} \geq \pi \geq \frac{n^{1/n}}{1 + n^{1/n}}.$$

**Proof.** Let  $(\sigma, \mu)$  be a separating equilibrium. This implies that the last bettor always follows his signal:  $\sigma_n(s^{n-1}; q^H) = H$  for all  $s^{n-1} \in S^{n-1}$ . In particular, (i)  $\sigma_n(\bar{H}, \dots, \bar{H}; q^H) = H$  and (ii)  $\sigma_n(H, \dots, H; q^H) = H$ .

(i) From Lemmas 6 and 7 we have ( $h = 0$ ):  $\mu_n(\theta_H | s^{n-1}; q^H) \geq \frac{h(s^{n-1})+1}{n+1} \Leftrightarrow \frac{\pi^{2-n}}{\pi^{2-n} + (1-\pi)^{2-n}} \geq \frac{1}{n+1} \Leftrightarrow \frac{n^{1/(n-2)}}{1+n^{1/(n-2)}} \geq \pi$ .

(ii) From Lemmas 6 and 7 we have ( $h = n - 1$ ):  $\mu_n(\theta_H | s^{n-1}; q^H) \geq \frac{h(s^{n-1})+1}{n+1} \Leftrightarrow \frac{\pi^n}{\pi^n + (1-\pi)^n} \geq \frac{n}{n+1} \Leftrightarrow \pi \geq \frac{n^{1/n}}{1+n^{1/n}}$ . This completes the proof. ■

This necessary condition for a separating equilibrium to exist becomes more and more restrictive as the number of bettors,  $n$ , increases.<sup>5</sup> Consequently, when the number of players increases, there is a clear distinction between the properties of the equilibria of the static and dynamic markets.

While a separating equilibrium generally fails to exist, some bettors may follow their own signal when the majority of previous decisions is not too overwhelming in favor of one of the outcomes. The informativeness of bets crucially depends on the observed history. It is possible that a combination of uninformativeness, in the sense that an individual's private information

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<sup>5</sup>It is not difficult to provide a necessary and *sufficient* condition for a separating equilibrium to exist. This condition is slightly stronger than the condition of the previous proposition but it is intractable (it cannot be analytically solved in  $\pi$  for arbitrary values of  $n$ ).

cannot be inferred from his action, bets (herd and contrarian bets) and informative bets may constitute an equilibrium. In the next subsection we characterize such equilibria in markets with two and three bettors.

### 4.3 Characterization of Equilibria with Two and Three Bettors

It can be shown that, in the two-period sequential game, the first player always bets in accordance with his signal. In contrast, the strategy of the bettor who takes his decision in the second position depends on the quality of bettors' privately held information. If the quality is low, then he always bets on the current longshot. If the information quality is high, then he always follows his signal. Details are provided in example 1.<sup>6</sup>

**Example 1** *Consider the 2-bettor sequential betting game. There exists a separating equilibrium if and only if  $\pi \geq \frac{\sqrt{2}}{1+\sqrt{2}}$ . Otherwise, there is an equilibrium where the first bettor always follows his signal and the second bets against the first bettor's choice whatever the content of his signal. Those equilibria are sequential equilibria (Kreps and Wilson, 1982), and for generic values of  $\pi$  there is no other sequential equilibrium.*

Hence, herd behavior is not possible with only two bettors. The reason is that the negative payoff externality from the first bet means that it is never profitable for the second bettor to follow the first bettor. When the parimutuel betting market involves more than two players, equilibrium herd behavior is possible. Past histories can overwhelm later bettors' signals if  $\pi$  is relatively high. Example 2 illustrates the point.

**Example 2** *Consider the 3-bettor sequential betting game. Assume that the first and the second bettors always follow their private signals.*

- If  $0.547 \simeq \pi^* < \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}$ ,<sup>7</sup> then  $\sigma$  is an equilibrium iff the third bettor uses a contrarian strategy. In other words,  $\sigma_3(H, H; q_3) = \bar{H}$  for all  $q_3 \in Q_3$  and  $\sigma_3(H, \bar{H}; q^H) = \sigma_3(\bar{H}, H; q^H) = H$  for all  $H \in \{A, B\}$ .
- If  $\frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} < \pi < 3/4$ , then  $\sigma$  is an equilibrium iff the third bettor always follows his signal, i.e.,  $\sigma$  is a separating strategy.
- If  $\pi > 3/4$ , then  $\sigma$  is an equilibrium iff the third bettor uses a herding strategy. That is,  $\sigma_3(H, H; q_3) = H$  for all  $q_3 \in Q_3$  and  $\sigma_3(H, \bar{H}; q^H) = \sigma_3(\bar{H}, H; q^H) = H$  for all  $H \in \{A, B\}$ .

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<sup>6</sup>The equilibrium characterizations of the two examples are simple, but cumbersome to prove because many cases have to be considered. The details can be found in Koessler and Ziegelmeier (2003, Propositions 6 and 7).

<sup>7</sup> $\pi^*$  solves  $\pi^2 = (1 - \pi)^2(2 - \pi)$ .

Contrary to the 2-bettor case, herd behavior can occur when the quality of information is high. The behavior of the third bettor in the equilibria described in Example 2 is relatively intuitive. If the quality of information is low, then the negative effect of the odds against the favorite dominates the belief that this horse will win, and thus the third bettor bets on the longshot whatever his private information. If the quality of information is high, then the third bettor strongly believes that the favorite will win, and thus he bets on the favorite and disregards his private information. For intermediate qualities of information, neither information externalities nor payoff externalities dominate, and thus the third bettor follows his signal, using a separating strategy.<sup>8</sup>

## 5 Conclusion

To the best of our knowledge, this paper is the first to investigate betting behavior in parimutuel markets under asymmetric information. Our key results (Propositions 1 and 2) lead us to conclude that there is a clear-cut distinction between the equilibrium properties of simultaneous and sequential betting markets. The odds reflect all private information in the equilibrium in the simultaneous case, but not necessarily in the sequential case. Informational inefficiencies in the sequential markets are due to (i) herd behavior which can arise with high quality of private information, and to (ii) contrarian behavior, which can occur with low quality of private information.

There are some interesting relationships between our framework and the literature on social learning. One parallel can be drawn between the analysis offered here and the literature on “information cascades”. An information cascade can be defined as a choice sequence, in which some agents act as if they ignore their private information and follow the choice other agents make earlier in the sequence (see Banerjee, 1992 and Bikhchandani et al., 1992). In these herding models, agents’ payoffs are not directly affected by the actions of others. In this respect, our work can be seen as an extension of rational herd behavior models, but in which negative payoff externalities are included. Even though long-run herding is impossible in sequential parimutuel betting markets, our results imply that informational inefficiency, at least in the short-run, is a feature of the parimutuel mechanism.<sup>9</sup>

Most of the literature on rational herding, with the notable exception of Avery and Zemsky (1998), assumes that prices of taking an action are fixed. The main differences between Avery

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<sup>8</sup>There are other equilibria than those described in Example 2. Unfortunately, those equilibria may coincide, i.e., different types of equilibria may exist for generic qualities of information. For example, it can be shown that if  $\pi \leq \frac{\sqrt{2}}{1+\sqrt{2}}$ , then there is an equilibrium where the first bettor follows his signal and the other bettors adopt contrarian behavior.

<sup>9</sup>This is consistent with the experimental studies of Koessler, Noussair, and Ziegelmeyer (2005) and Plott, Witt, and Yang (2003), which demonstrate that even though the parimutuel institution is a powerful device for aggregating information, phenomena consistent with herding and contrarian behavior are observed.

and Zemsky's (1998) asset pricing model and a parimutuel betting market are the following. First, in Avery and Zemsky's model, the price is determined by a market maker according to his information about past trades, while in parimutuel betting markets the price mechanism is exogenous and ensures that average bettors' return is null or negative (if transaction costs are strictly positive). Second, in a parimutuel betting market, the return for each player depends on his expectation about the behavior of later participants. Hence, in such markets, the analysis is greatly complicated by the fact that fully rational agents are concerned not only with learning from predecessors, but also with signaling to successors.<sup>10</sup>

This intractable difficulty may be avoided by modeling agents as myopic price takers, who treat the current odds, at the time they make their bets, as the final odds. This latter framework is analyzed in Koessler and Ziegelmeier (2003), and is closely related to the environment of Avery and Zemsky. More precisely, if we assume that bettor  $i$  considers the betting odds as fixed once he makes his decision, then his perceived utility function is given by  $u_i(s^i, \theta_H) = O_H(s^{i-1}) + 1$  if  $s_i = H$  and  $u_i(s^i, \theta_H) = 0$  if  $s_i \neq H$ . If, in addition, the odds against each horse  $H$  are given by  $O_H(s^{i-1}) = \frac{1 - \Pr(\theta_H | s^{i-1})}{\Pr(\theta_H | s^{i-1})}$  then the implicit price of horse  $H$  is equal to  $\frac{1}{O_H(s^i) + 1} = \Pr(\theta_H | s^{i-1})$ .<sup>11</sup> This is exactly the asset price in the model of Avery and Zemsky (1998).

## Appendix

**Proof of Lemma 3.** Assume that  $k < n$  bettors always follow their signal,  $a$  bettors always bet on horse  $A$ , and  $b = n - a - k$  bettors always bet on horse  $B$ . Let  $N_K$  be the set of bettors who follow their signals,  $N_A$  the set of bettors who always bet on horse  $A$ , and  $N_B$  the set of bettors who always bet on horse  $B$ . For each bettor  $i \in N_K$  (assuming  $k \neq 0$ ) we have

$$U_i(A, \sigma_{-i} | q^A) = n\pi \sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{a+1+j}, \quad U_i(B, \sigma_{-i} | q^A) = n(1-\pi) \sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{b+1+j}$$

$$U_i(A, \sigma_{-i} | q^B) = n(1-\pi) \sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{a+1+j}, \quad U_i(B, \sigma_{-i} | q^B) = n\pi \sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{b+1+j}.$$

<sup>10</sup>Dasgupta (2000) performs a similar analysis with the notable difference that he considers positive payoff externalities with an additional requirement of complete agreement on investment decisions. See also Corsetti, Dasgupta, Morris, and Shin (2004) for a 2-period model.

<sup>11</sup>The implicit price of horse  $H$  corresponds to the price of obtaining a claim to one unit of money in the event that horse  $H$  wins the race and it is referred to as the subjective probability of horse  $H$  in the parimutuel literature.

For each bettor  $i \in N_A$  we have

$$U_i(A, \sigma_{-i} | q^A) = n\pi \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+j}, \quad U_i(B, \sigma_{-i} | q^A) = n(1-\pi) \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+1+j}$$

$$U_i(A, \sigma_{-i} | q^B) = n(1-\pi) \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+j}, \quad U_i(B, \sigma_{-i} | q^B) = n\pi \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+1+j}.$$

Finally, for each bettor  $i \in N_B$  we have

$$U_i(A, \sigma_{-i} | q^A) = n\pi \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+1+j}, \quad U_i(B, \sigma_{-i} | q^A) = n(1-\pi) \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+j}$$

$$U_i(A, \sigma_{-i} | q^B) = n(1-\pi) \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+1+j}, \quad U_i(B, \sigma_{-i} | q^B) = n\pi \sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+j}.$$

It is easy to verify that each bettor  $i \in N_A$  is rational only if  $a < b+1$  and each bettor  $i \in N_B$  is rational only if  $b < a+1$ , which implies that  $b-1 < a < b+1$ , i.e.,  $a = b = \frac{n-k}{2}$ . In this case, it is rational for each bettor  $i \in N_K$  to follow his signal. Moreover, each bettor  $i \in N_H$  rationally follows his signal when he receives the signal  $q^H$ . It remains to check under which conditions each bettor  $i \in N_H$  bets on  $H$  even when he receives the signal  $q^{\bar{H}}$ . From the expected utilities given above, this condition is satisfied if and only if inequality (2) on page 6 is satisfied, with  $0 \leq k < n$ . The fact that there are exactly  $C_n^k \times C_{\frac{n-k}{2}}^{\frac{n-k}{2}}$  partially separating equilibria with  $k$  separating strategies is obvious. ■

**Proof of Lemma 4.** To show that values of  $\pi$  satisfying the condition for a  $k$ -separating equilibrium to exist belong to an interval  $[1/2, \pi(k, n)]$ , where  $\pi(k, n) < 1$ , we simplify inequality (2), using lemma 8. Then, we present useful properties of this new formulation in lemmas 9 and 10 that are sufficient to prove Lemma 4.

**Lemma 8** *Inequality (2) is satisfied if and only if  $g(k, n, \pi) \leq 0$ , where*

$$g(k, n, \pi) = \sum_{j=0}^{k+1} \left( \frac{\pi}{1-\pi} \right)^j C_{k+1}^j \frac{k-2j+1}{k-2j-n}.$$

**Proof.** Let  $f(k, n, \pi) = \frac{\pi}{1-\pi} - \frac{\sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+j}}{\sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+1+j}}$ . We have to show that  $f(k, n, \pi) \leq 0$  is equivalent to  $g(k, n, \pi) \leq 0$ . Let  $x = \frac{\pi}{1-\pi} \in (1, +\infty)$  and  $a = \frac{n-k}{2}$ . We have  $f(k, n, \pi) \leq 0 \Leftrightarrow x \leq \frac{\sum_{j=0}^k \frac{C_k^j x^j}{a+j}}{\sum_{j=0}^k \frac{C_k^j x^j}{a+j+1}} \Leftrightarrow \sum_{j=-1}^k \frac{C_k^j x^{j+1}}{a+j+1} \leq \sum_{j=0}^{k+1} \frac{C_k^j x^j}{a+j}$ , since by convention  $C_k^{-1} = C_k^{k+1} = 0$ . The last inequality is equivalent to  $\sum_{j=0}^{k+1} \frac{x^j}{a+j} (C_k^j - C_k^{j-1}) \geq 0$ . Using the definition of the binomial

coefficient, it is not difficult to show that  $C_k^j - C_k^{j-1} = C_{k+1}^j \frac{k+1-2j}{k+1}$ . Substituting this value into the last inequality, we get  $\sum_{j=0}^{k+1} \frac{x^j}{a+j} C_{k+1}^j \frac{k+1-2j}{k+1} \geq 0 \Leftrightarrow \sum_{j=0}^{k+1} x^j C_{k+1}^j \frac{k+1-2j}{(\frac{n-k}{2}+j)(k+1)} \geq 0 \Leftrightarrow \sum_{j=0}^{k+1} x^j C_{k+1}^j \frac{k+1-2j}{n-k+2j} \geq 0 \Leftrightarrow g(k, n, \pi) \leq 0$ . ■

**Lemma 9** For all  $n \geq 2$  and  $k < n$ , there exists  $\pi, \pi' \in (1/2, 1)$  such that  $g(k, n, \pi) < 0$  and  $g(k, n, \pi') > 0$ .

**Proof.** It is equivalent to prove this property for  $f(k, n, \pi)$ . We have

$$f(k, n, 1/2) = 1 - \frac{\sum_{j=0}^k \frac{1}{j!(k-j)!(\frac{n-k}{2}+j)}}{\sum_{j=0}^k \frac{1}{j!(k-j)!(\frac{n-k}{2}+j+1)}} < 0.$$

Moreover,  $\lim_{\pi \rightarrow 1^-} \frac{\sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+j}}{\sum_{j=0}^k \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+j+1}} = \frac{\frac{n-k}{2}+k+1}{\frac{n-k}{2}+k}$ , and  $\lim_{\pi \rightarrow 1^-} \frac{\pi}{1-\pi} = +\infty$ , which implies that  $\lim_{\pi \rightarrow 1^-} f(k, n, \pi) > 0$  for all  $n$  and  $k$ . Hence, there exists  $\pi, \pi' \in (1/2, 1)$ , such that  $f(k, n, \pi) < 0$  and  $f(k, n, \pi') > 0$ . ■

**Lemma 10** For all  $n \geq 2$  and  $k < n$ ,  $g(k, n, \cdot)$  is strictly increasing in  $\pi$  on the interval  $(1/2, 1)$ .

**Proof.** We have to show that  $\frac{\partial g(k, n, \pi)}{\partial \pi} > 0$  for all  $\pi \in (1/2, 1)$ . Let  $x = \frac{\pi}{1-\pi} \in (1, +\infty)$ . We have  $\frac{\partial g(k, n, \pi)}{\partial \pi} > 0 \Leftrightarrow \sum_{j=0}^{k+1} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{k-2j-n} > 0$ . Assume that  $k$  is an even number. Then, the last inequality is equivalent to

$$\begin{aligned} & \sum_{j=0}^{k/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=(k/2)+1}^{k+1} j x^{j-1} C_{k+1}^j \frac{2j-k-1}{n+2j-k} \\ \Leftrightarrow & \sum_{j=0}^{k/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{k/2} (k+1-j) x^{k-j} C_{k+1}^{k+1-j} \frac{2(k+1-j)-k-1}{n+2(k+1-j)-k} \\ \Leftrightarrow & \sum_{j=0}^{k/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{k/2} (k+1-j) x^{k-j} C_{k+1}^j \frac{k-2j+1}{n-2j+k+2}, \end{aligned}$$

because  $C_{k+1}^j = C_{k+1}^{k+1-j}$ . To prove the last inequality, it is sufficient to show that for all  $j = 0, 1, \dots, k/2$  we have  $j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < (k+1-j) x^{k-j} C_{k+1}^j \frac{k-2j+1}{n-2j+k+2}$ . Since  $x > 1 \Rightarrow x^{j-1} < x^{k-j}$  for all  $j = 0, 1, \dots, k/2$ , it is sufficient to show that  $j \frac{k-2j+1}{n+2j-k} \leq (k+1-j) \frac{k-2j+1}{n-2j+k+2}$  or, equivalently,  $k+1-2j \geq 0$ , which is satisfied for all  $j = 0, 1, \dots, k/2$ . Now, we assume

that  $k$  is an odd number. Then, we have

$$\begin{aligned}
& \sum_{j=0}^{(k-1)/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=(k+1)/2}^{k+1} j x^{j-1} C_{k+1}^j \frac{2j-k-1}{n+2j-k} \\
\Leftrightarrow & \sum_{j=0}^{(k-1)/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=(k+3)/2}^{k+1} j x^{j-1} C_{k+1}^j \frac{2j-k-1}{n+2j-k} \\
\Leftrightarrow & \sum_{j=0}^{(k-1)/2} j x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{(k-1)/2} (k+1-j) x^{k-j} C_{k+1}^{k+1-j} \frac{k-2j+1}{n-2j+k+2}.
\end{aligned}$$

The rest of the proof is as before. ■

**Proof of Proposition 1.** We show that when the number of bettors tends to infinity, the equilibrium condition for a bettor who does not use the separating strategy is not satisfied. Consider a sequence of  $k_n$ -equilibria,  $n \geq 2$ , and assume by way of contradiction that there is an infinite subsequence  $(k_n)_n$  with  $k_n > 0$  for all  $n$ , i.e., an infinite subsequence of equilibria that are not separating. Consider, without loss of generality, a player who always bets on  $A$ . For this player, denoted by  $i$ , the expected payoff for choosing  $A$  must be higher than the expected payoff for choosing  $B$  even when this player has the signal  $q_i = q^B$ :  $U_i(A, \sigma_{-i} | q^B) \geq U_i(B, \sigma_{-i} | q^A)$ . By Lemma 3 we know that the number of bettors who always bet on  $A$  is the same as the number of bettors who always bet on  $B$ , i.e., it is equal to  $\frac{n-k_n}{2}$ . Hence, the last inequality is equivalent to

$$(1-\pi) \mathbb{E} \left[ \frac{n}{\frac{n-k_n}{2} + \sum_{j=1}^{k_n} \mathbf{1}_{[q_j=q^A]}} \mid \theta_A \right] \geq \pi \mathbb{E} \left[ \frac{n}{1 + \frac{n-k_n}{2} + \sum_{j=1}^{k_n} \mathbf{1}_{[q_j=q^B]}} \mid \theta_B \right] \quad (5)$$

If  $(k_n)_n$  is bounded, then when  $n \rightarrow \infty$  this condition becomes  $2(1-\pi) \geq 2\pi$ , a contradiction. If  $(k_n)_n$  is not bounded, then we can extract a strictly increasing subsequence, for which condition (5) becomes, for  $n \rightarrow \infty$ ,

$$(1-\pi) \frac{n}{\frac{n-k_n}{2} + \pi k_n} \geq \pi \frac{n}{1 + \frac{n-k_n}{2} + \pi k_n} \Leftrightarrow (1-\pi) \frac{2n}{n + k_n(2\pi - 1)} \geq \pi \frac{2n}{2 + n + k_n(2\pi - 1)},$$

which also yields the contradiction  $1-\pi > \pi$  when  $n \rightarrow \infty$ . Therefore, for  $n$  sufficiently large, we know that there is no  $k_n$ -separating equilibrium with  $k_n > 0$ , which proves the proposition. ■



## References

- AVERY, C. AND P. ZEMSKY (1998): “Multidimensional Uncertainty and Herd Behavior in Financial Markets,” *American Economic Review*, 88, 724–748.
- BANERJEE, A. (1992): “A Simple Model of Herd Behavior,” *Quarterly Journal of Economics*, 107, 797–818.
- BIKHCHANDANI, S., D. HIRSHLEIFER, AND I. WELCH (1992): “A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades,” *Journal of Political Economy*, 100, 992–1026.
- CHADHA, S. AND R. E. QUANDT (1996): “Betting Bias and Market Equilibrium in Racetrack Betting,” *Applied Financial Economics*, 6, 287–292.
- CORSETTI, G., A. DASGUPTA, S. MORRIS, AND H. S. SHIN (2004): “Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders,” *Review of Economic Studies*, 71, 87–113.
- DASGUPTA, A. (2000): “Social Learning with Payoff Complementarities,” mimeo, Yale University.
- FEENEY, R. AND S. P. KING (2001): “Sequential Parimutuel Games,” *Economics Letters*, 72, 165–173.
- HURLEY, W. AND L. McDONOUGH (1995): “A Note on the Hayek Hypothesis and the Favorite–Longshot Bias in Parimutuel Betting,” *American Economic Review*, 85, 949–955.
- KOESSLER, F., C. NOUSSAIR, AND A. ZIEGELMEYER (2005): “Individual Behavior and Beliefs in Parimutuel Betting Markets,” mimeo.
- KOESSLER, F. AND A. ZIEGELMEYER (2003): “Parimutuel Betting under Asymmetric Information,” Discussion Paper on Strategic Interaction 2003-34, Max-Planck Institute of Economics, Jena.
- KOESSLER, F., A. ZIEGELMEYER, AND M.-H. BROIHANNE (2003): “The Favorite-Longshot Bias in Sequential Parimutuel Betting with Non-Expected Utility Players,” *Theory and Decision*, 54, 231–248.
- KREPS, D. M. AND R. WILSON (1982): “Sequential Equilibria,” *Econometrica*, 50, 863–894.
- OTTAVIANI, M. AND P. N. SØRENSEN (2005): “The Timing of Bets and the Favorite-Longshot Bias,” mimeo.

- PLOTT, C. R., J. WITT, AND W. C. YANG (2003): “Parimutuel Betting Markets as Information Aggregation Devices: Experimental Results,” *Economic Theory*, 22, 311–351.
- SAUER, R. D. (1998): “The Economics of Wagering Markets,” *Journal of Economic Literature*, 36, 2021–2064.
- TERRELL, D. AND A. FARMER (1996): “Optimal Betting and Efficiency in Parimutuel Betting Markets with Information Costs,” *The Economic Journal*, 106, 846–868.
- VAUGHAN WILLIAMS, L. (1999): “Information Efficiency in Betting Markets: A Survey,” *Bulletin of Economic Research*, 51, 1–30.
- WATANABE, T. (1997): “A Parimutuel System with two Horses and a Continuum of Bettors,” *Journal of Mathematical Economics*, 28, 85–100.
- WATANABE, T., H. NONOYAMA, AND M. MORI (1994): “A Model of a General Parimutuel System: Characterizations and Equilibrium Selection,” *International Journal of Game Theory*, 23, 237–260.