A Note on Quantal Response Equilibra in Bargaining

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Abstract

The Nash Bargaining problem in the context of a random utility model yields a stochastic demand for each player, conditional on his or her beliefs regarding the other player's behavior. We derive a symmetric logit equilibrium under naive expectations that converges to the Nash axiomatic solution as noise in utility vanishes. A numerical approximation to the symmetric logit equilibrium under rational expectations (Quantal Response Equilibrium) is also computed.

1 Introduction

This paper examines bargaining behavior within the framework of a random utility model. We focus on the usual Nash demand game in which two players, $i = 1, 2$, bargain about how to split a fixed surplus of size 1. Both players simultaneously demand the respective shares $x^1, x^2 \in [0, 1]$ of the surplus they want to keep. If $x^1 + x^2 \leq 1$, demands are said to be compatible, and each player obtains what he or she demanded. In case that both demands add up to less than one, the residual $1 - x^1 - x^2$ vanishes. If $x^1 + x^2 > 1$, bargaining fails and both players end up with zero. This means that player i's material payoff is given by

$$\pi^i(x^i, x^j) = \begin{cases} x^i & \text{, if } x^i + x^j \leq 1 \quad (j \neq i) \\ 0 & \text{, otherwise.} \end{cases} \tag{1}$$

As is well known, this is a coordination game with a continuum of equilibria both in pure and mixed strategies. More specifically, all demand profiles (x^1, x^2) such that $x^1 + x^2 = 1$ are efficient Nash equilibria. Also, an infinite number of symmetric equilibria in mixed strategies exist such that, for all $\epsilon \in (0, \frac{1}{2}]$, the

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probability that player i will demand less than x is given by the distribution[1]

$$\Pr(x^i \leq x) \equiv F_\epsilon(x) = \begin{cases} 0 & , x < \epsilon \\ \frac{\epsilon}{1-x} & , \epsilon \leq x \leq 1-\epsilon \\ 1 & , x > 1-\epsilon. \end{cases}$$

Of course, one can obtain a unique solution by imposing normative requirements on the players' behavior and on the outcome of the bargaining process (see, e.g., Nash, 1950; Nash, 1953; Kalai and Smorodinsky, 1975) However, whether a normative approach is a good method for predicting actual behavior is an open question. Already in its seminal contribution, Nash (1953) recognized the desirability of deriving any axiomatic solution as the limit equilibrium of a sequence of perturbed non-cooperative games.[2]

In one of the first examples of an equilibrium refinement concept, Nash (1953) suggested a "smoothed" version of the bargaining game. For this purpose, he introduced a nonnegative random variable ε with expected value $\overline{\varepsilon} < \infty$.[3] One can think of ε as an uncertain additional surplus (in excess to the sure unit pie) that may render demands outside the set $B = \{(d^1, d^2) : d^1 + d^2 \leq 1\}$ compatible, so that for $i = 1, 2$,

$$\pi^i(x^i, x^j|\varepsilon) = \begin{cases} x^i & , \text{if } x^i + x^j \leq 1 + \varepsilon \quad (j \neq i) \\ 0 & , \text{otherwise.} \end{cases} \tag{2}$$

Thus, player i's expected material payoff from demand profile (x^i, x^j) in the smoothed game can be written as $\mathsf{E}\pi^i(x^i, x^j|\varepsilon) = x^i h(\varepsilon|x^i, x^j)$, where $h(\varepsilon|x^i, x^j) \equiv 1 - \Pr(\varepsilon \leq x^i + x^j - 1)$. Nash (1953) showed that the bargaining solution $(x^i_N, x^j_N) = \arg\max_{(x^i,x^j)\in B} x^i x^j$ is indeed the limit of any sequence of equilibrium demands $(x^i(\overline{\varepsilon}), x^j(\overline{\varepsilon}))$ generated from the perturbed game as $\overline{\varepsilon} \to 0$.

While the perturbed game (2) assumes a random shock on the size of the surplus, which is common to both players, in this paper we follow an alternative route, assuming that individual utility functions are subject to random perturbations which are independent of each other. This kind of individual payoff perturbations has been used by Harsanyi (1973) to purify mixed-strategy equilibria, and together with strategy trembles (Selten, 1975) are common to all equilibrium selection attempts, a field of research inspired by Nash (see, e.g., Harsanyi and Selten, 1988; Güth and Kalkofen, 1989).

[1] In such equilibrium, each player demands a share $x^i = \epsilon$ with mass probability $\frac{\epsilon}{1-\epsilon}$, whereas demands in the interval $(\epsilon, 1-\epsilon]$ are chosen according to the probability density function $f_\epsilon(x) = \frac{\epsilon}{(1-x)^2}$. Hence, the expected utility of choosing $x^i \in [0,1]$ is $u^i_e(x^i) = x^i \cdot F_\epsilon(1-x^i) = \epsilon$ if $x^i \in [\epsilon, 1-\epsilon]$, and $u^i_e(x^i) < \epsilon$ otherwise.

[2] A sequence of mixed-strategy equilibria $\{F_\epsilon\}$ trivially converges to the equal split solution as $\epsilon \to \frac{1}{2}$, i.e., as the support of F_ϵ vanishes.

[3] Nash's analysis is not restricted to bargaining games with transferable utility. In its original version, the smoothed game proposed by Nash (1953) allows for ε to depend on (x^i, x^j). He assumed correlated random shocks in the sense of $\varepsilon^i(x^i, x^j) = \varepsilon^j(x^j, x^i)$.

2 A random utility model

Consider a modified version of the Nash bargaining game in which each player's preferences are represented by a utility function that includes an additive noise process. In particular, let

$$u^i(x^i, x^j) = \pi^i(x^i, x^j) + \varepsilon^i(x^i) \quad (j \neq i), \tag{3}$$

where $\{\varepsilon^i(x^i)\}$ is assumed to be a continuous stationary max-stable process such that, for each $x^i \in [0,1]$, $\varepsilon^i(x^i) \sim iid$ Type I Extreme Value with $\mathsf{Var}\ \varepsilon^i(x^i) = 1/\lambda^2$. Resnick and Roy (1991) provide a behavioral interpretation of (3) as being generated by a collection of arguments for and against each possible choice of x_i, which player i processes "in a boundedly rational fashion, recalling only the most [favorable] signal for each alternative" (p. 273).

While the particular realization of $\varepsilon^i(\cdot)$ is player i's private information, its underlying distribution is assumed to be commonly known. In this way, the bargaining game becomes a perturbed game of incomplete information in which player i's type can be thought of as a particular realization of $\varepsilon^i(\cdot)$. Notice that the distributional assumptions regarding $\varepsilon^i(\cdot)$ imply that, given player j's demand, x^j, the probability of player $i \neq j$ being indifferent between any two demands $x^i, y^i \in [0,1]$, $x^i \neq y^i$, is zero.

2.1 Logit equilibrium under naive expectations

Suppose for a moment that player i (naively) believes that his partner j will make a fixed demand $x^j = b^i$ with probability one. In this case, the material payoff that player i (naively) expects from each feasible demand $x^i \in [0,1]$ he or she can choose is

$$\pi^i(x^i|b^i) = \begin{cases} x^i & \text{, if } x^i \leq 1 - b^i \\ 0 & \text{, otherwise.} \end{cases} \tag{4}$$

According to the random utility model (3), however, player i's preferences not only depend on the expected material payoff (4), but also on the realization of the type $\varepsilon^i(\cdot)$. Moreover, given the distributional assumptions regarding $\varepsilon^i(\cdot)$, there will be *almost surely* a unique alternative $x^i \in [0,1]$ for each type of player holding beliefs b^i that maximizes expected utility. Following Ben-Akiva, Litinas and Tsunokawa (1985), this leads to a continuous version of the logit choice model, which allows an exogenous observer to express ex ante choice probabilities via the density function

$$f(x^i|\lambda, b^i) = \frac{\exp\{\lambda\pi^i(x^i|b^i)\}}{\int_0^1 \exp\{\lambda\pi^i(y|b^i)\}dy}, \qquad x^i \in [0,1]. \tag{5}$$

More specifically, substituting (4) into (5), we obtain

$$f(x^i|\lambda, b^i) = \begin{cases} K(\lambda, b^i)\exp\{\lambda x^i\} & \text{, if } x^i \leq 1 - b^i \\ \\ K(\lambda, b^i) & \text{, otherwise,} \end{cases} \tag{6}$$

where $K(\lambda, b^i) \equiv \lambda \left[\exp\{\lambda(1-b^i)\} - 1 + \lambda b^i\right]^{-1}$.

A minimum requirement on player i's beliefs is that they match the *expected value* of his opponent's actual behavior (as described by the density f^j on [0,1] of j's optimal demand, which is in turn generated by the distribution of types $\varepsilon^j(\cdot)$). This is a very mild consistency condition, since it does not impose "rational expectations" on second and higher moments. In this sense, if players i and j are ex ante symmetric, a "naive equilibrium" can be defined as follows:

Definition 1 (Logit equilibrium under naive expectations)
For any fixed value $0 < \lambda < \infty$, a logit equilibrium under naive expectations is uniquely given by $f^{naive}(x|\lambda) = f(x|\lambda, b^*(\lambda))$, where the naive equilibrium belief $b^*(\lambda) < \frac{1}{2}$ is the fix point of

$$
\begin{aligned}
b^* &= \int_0^1 x\ f(x|\lambda, b^*)dx \\
&= \frac{\frac{1}{\lambda} + \lambda\left(b^* - \frac{b^{*2}}{2}\right) + \left(1 - b^* - \frac{1}{\lambda}\right)\exp\{\lambda(1-b^*)\}}{\exp\{\lambda(1-b^*)\} - 1 + \lambda b^*}.
\end{aligned}
$$

The density $f^{naive}(x^i|\lambda)$ is depicted in Figure 1 for various values of the precision parameter λ. Notice that the likelihood of making any particular demand increases exponentially up to $1 - b^*(\lambda)$, and declines abruptly there to remain at a minimum level for all $x^i > 1 - b^*(\lambda)$.

Notice also that $b^*(\lambda) \to \frac{1}{2}$ as $\lambda \to \infty$. This convergence of naive equilibrium beliefs to the Nash-solution is illustrated in Figure 2. In other words, although the resulting density $f^{naive}(x^i|\lambda)$ has full support on $[0, 1]$ it approaches the degenerate distribution that assigns probability one to the equal-split solution as the random processes $\varepsilon^i(\cdot)$, $i = 1, 2$ vanish.

2.2 Approximate equilibrium under rational expectations

The logit naive equilibrium outlined in the previous section, $f^{naive}(x|\lambda)$, is generated by beliefs that put probability one on a single, fixed value $x^j = b^*(\lambda) \in [0, 1]$ ($j \neq i$). In contrast, a "rational expectations" equilibrium as defined by McKelvey and Palfrey (1995) in the context of random utility models, requires each player's beliefs to be consistent with the true distribution of his or her partner's actual behavior. In other words, player i's beliefs should assign the same probability measure on all $x^j \in [0, 1]$ as the actual ex ante distribution of player j's possible choices.

Assume the existence of a *symmetric* quantal response equilibrium under rational expectations, such that the ex ante probability of player $i = 1, 2$ demanding a share smaller than $x \in [0, 1]$ is given by the cumulative distribution function $F(x|\lambda) \equiv \Pr(x^i \leq x|\lambda)$, with $F(0|\lambda) = 0$ and $F(1|\lambda) = 1$ for all $0 < \lambda < \infty$, and associated density function $f(x|\lambda)$. Then, imposing rational expectations means that, for every $x^i \in [0,1]$, player i's monetary income can be written as

$$\pi^i(x^i|\lambda) = \begin{cases} x^i & \text{, with probability } F(1-x^i|\lambda) \\ 0 & \text{, with probability } 1-F(1-x^i|\lambda). \end{cases} \tag{7}$$

Thus, for every $x^i \in [0,1]$, player i's equilibrium beliefs about his or her own expected monetary income must be

$$\pi^i_e(x^i|\lambda) \equiv \mathsf{E}\pi^i(x^i|\lambda) = x^i \cdot F(1-x^i|\lambda).$$

In analogy to (3) this means that, in equilibrium, the expected utility of player i (i.e., the expected utility from each $x^i \in [0,1]$ given his or her idiosyncratic type $\varepsilon^i(\cdot)$) is equal to

$$u^i(x^i|\lambda) = \pi^i_e(x^i|\lambda) + \varepsilon^i(x^i).$$

This, in turn, yields almost surely a dominant strategy for each realization of $\varepsilon^i(\cdot)$ which, from the perspective of an outside observer (including i's opponent), can be characterized in the following way:

Definition 2 (Logit equilibrium under rational expectations)
For any fixed value $0 < \lambda < \infty$, a logit equilibrium under rational expectations (or logit quantal response equilibrium) is given by

$$f^*(x^i|\lambda) = K(\lambda)\exp\left\{\lambda x^i \cdot F^*(1-x^i|\lambda)\right\}. \tag{8}$$

At this point, it is convenient to summarize some implications of (8) regarding $f^*(x^i|\lambda)$:

1. $f^*(0|\lambda) = K(\lambda) = f^*(1|\lambda)$, where the second equality follows from the fact that $F^*(0|\lambda) = 0$.

2. $F^*(\cdot|\lambda)$ has full support since, by definition, $F^*(x|\lambda) \in [0,1]$, which implies that $\lambda x F^*(1-x|\lambda) \geq 0$ or, equivalently, $f^*(x|\lambda) \geq K(\lambda)$.

3. $f^*(x|\lambda) > K(\lambda)$ for at least some $x \in (0,1)$. To see this, notice that if it were not the case, then $f^*(x|\lambda) = K(\lambda) = 1$ (since $F^*(1|\lambda) = 1$), so that $x(1-x) = 0$ for all $x \in [0,1]$, a contradiction.

It is beyond the scope of this paper to obtain a closed expression for $F^*(x|\lambda)$. This is not an easy task, since there might be multiple solutions. However, one approach to examine the kind of equilibrium behavior that can plausibly result from the perturbed game (3) is to find functions that *approximately* fulfill (8).

As noted by McLachlan and Peel (2001), p.3, a finite mixture of normal heteroscedastic components is a flexible and parsimonious way to model skewed (and possibly multimodal) distributions. Thus, if we conjecture the existence of a continuous equilibrium distribution $F^*(x|\lambda)$, one simple way to find an approximation to it is by fitting a finite mixture of (truncated) normal distributions on $[0,1]$.

More specifically, let $\Phi(\cdot;\mu,\sigma^2)$ denote the truncated normal cumulative distribution on $[0,1]$, with location and scale parameters μ and σ^2. Then, for a given grid $X(n) = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1)$, a best approximation to $F^*(x|\lambda)$ can be defined as the distribution $G^*(x;\widehat{\theta}(\lambda))$ (with associated density $g^*(x;\widehat{\theta}(\lambda))$) that minimizes the mean squared deviance

$$MSD(n,\widehat{\theta}|\lambda) \equiv \frac{1}{n+1} \sum_{x_k \in X(n)} \left[\ln g^*(x_k;\widehat{\theta}) - \ln g^*(0;\widehat{\theta}) - \lambda x_k \cdot G^*(1-x;\widehat{\theta}) \right]^2,$$

where $\widehat{\theta} = (\mu_1, \sigma_1, \mu_2, \sigma_2, p)$ and

$$G^*(x;\widehat{\theta}) \equiv p \cdot \Phi(x;\mu_1,\sigma_1^2) + (1-p) \cdot \Phi(x;\mu_2,\sigma_2^2).$$

Table 1 shows the the values $\widehat{\theta}(\lambda)$ that minimize the mean squared deviance $MSD(n,\theta(\lambda))$ when $n = 10000$. The resulting approximations $g^*(x|\widehat{\theta}(\lambda))$ are plotted in Figure 3. It is clear that an equilibrium distribution closely resembles a uni-modal, continuous distribution with a slightly positive skew. On the other hand, there is also some indication that equilibrium behavior tends to be more concentrated about the equal-split demand $x^i = \frac{1}{2}$ as λ increases. Visual inspection of Figure 3 suggests that a finite mixture of two truncated normals is good approximation to the $f^*(x|\lambda)$ for lower values of the precision parameter λ.

3 Conclusions

This paper shows how the Nash Bargaining problem in the context of a random utility model with Type-I Extreme Value disturbances yields a logit choice function for each player's demand, conditional on his or her beliefs regarding the other player's behavior. We derive a symmetric logit equilibrium under first-moment consistency of beliefs, which converges to the Nash axiomatic solution as noise in utility vanishes. The symmetric logit equilibrium under fully rational expectations (Quantal Response Equilibrium) is shown to be the solution of a non-linear differential equation, which can be numerically approximated as a mixture of two truncated normal distributions when the random disturbance term exhibits a relatively large variance. As this variance decreases, the equilibrium distribution of individual demands is more concentrated near the equal-split solution.

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Appendix: Tables and Figures

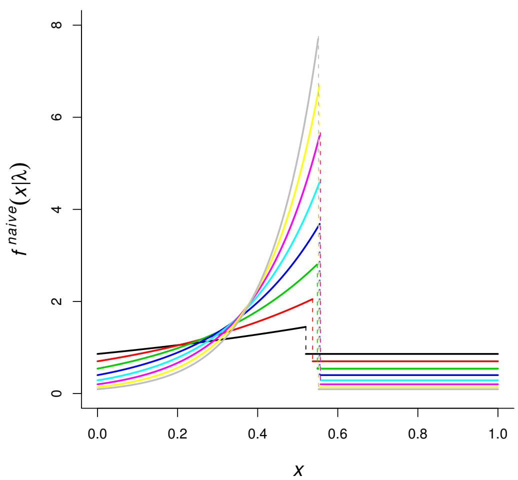

Figure 1: Naive-expectations logit choice densities $f^{naive}(x^i|\lambda)$, for $\lambda = 1, \ldots, 8$

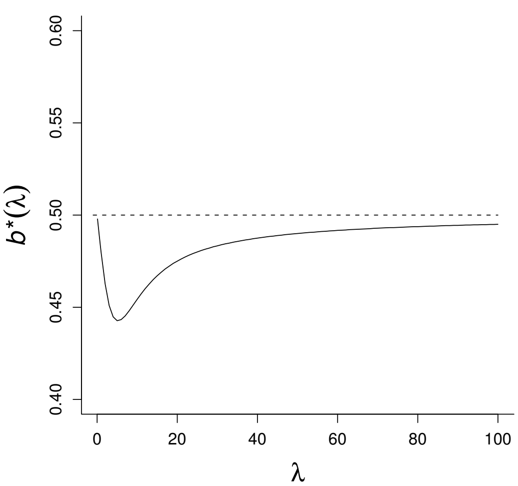

Figure 2: Naive equilibrium beliefs, $b^*(\lambda)$

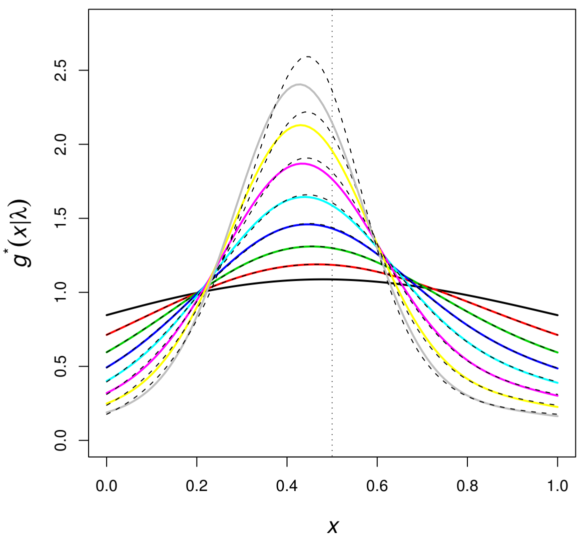

Figure 3: Finite mixture approximations $g^*(x|\widehat{\theta}(\lambda))$ to the rational-expectations logit choice densities $f^*(x^i|\lambda)$, for $\lambda = 1, \ldots, 8$. The dotted lines correspond to $g^*(0; \widehat{\theta}(\lambda)) \exp\left\{\lambda x_k \cdot G^*(1-x; \widehat{\theta}(\lambda))\right\}$.

λ	μ_1	σ_1	μ_2	σ_2	p	$1-p$	MSD
1	0.427	0.43	0.57	0.94	0.24	0.76	5×10^{-09}
2	0.433	0.33	0.61	0.76	0.37	0.63	1×10^{-07}
3	0.433	0.26	0.59	0.63	0.41	0.59	2×10^{-06}
4	0.432	0.22	0.57	0.56	0.46	0.54	1×10^{-05}
5	0.431	0.19	0.55	0.51	0.53	0.47	7×10^{-05}
6	0.430	0.17	0.53	0.49	0.60	0.40	3×10^{-04}
7	0.428	0.16	0.51	0.47	0.68	0.32	1×10^{-03}
8	0.427	0.14	0.50	0.47	0.76	0.24	3×10^{-03}

Table 1: Parameters of finite mixture approximation $G^*(x|\lambda)$ to the equilibrium distribution $F^*(x|\lambda)$