Evolution of division rules

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Abstract

Several *division rules* have been proposed in the literature regarding how an arbiter should divide a bankrupt estate. Different rules satisfy different sets of axioms, but all rules satisfy *claims boundedness* which requires that no contributor be given more than her initial contribution. This paper takes two non-cooperative bargaining games - the contracting game (Young, 1998a), and the Nash demand game, and adds the axiom of claims boundedness to the rules of these games. Outcomes prescribed by all the division rules are strict Nash equilibria in the one-shot version of both these augmented games. We show that the division suggested by the *truncated claims proportional rule* is the unique long run outcome if we embed the *augmented contracting game* in Young's (1993b) evolutionary bargaining model. With the *augmented Nash demand game* as the underlying bargaining game, the long run outcome is the division prescribed by the *constrained equal awards rule*.

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1. Introduction

"...the unjust is what violates the proportion; for the proportional is intermediate, and the just is proportional."

Nicomachean Ethics, Aristotle

In 'The Republic', Plato considers the problem of giving a flute to one of four children. Should it go to the child with the fewest number of toys, to the one who accidently found it, to the one who repaired it, or to the one who can play it best? Should a computer server with finite capacity first serve the smallest process, or the largest process? In case of organ transplant, should priority be given to those who will not survive without it, or to those who will survive the longest with it? Should the lowest claimants be given priority in bankruptcy settlements, or the highest claimants?[1] Several problems have a similar structure in which the claims of different individuals on the scarce resource differ on the basis of needs, merit, or/and rights, ex-ante contracts are absent,[2] and thus a mechanism for ex-post division is required.

This paper deals only with division problems in which the claims of the agents can be meaningfully measured, the resource to be allocated is divisible, and falls short of satisfying the claims of all agents completely. We find two approaches in the literature dealing with the ex-post division of limited resources (see Aumann and Maschler, 1985, for a historical account of a division problem from the Talmud, and Thomson, 2004, for a comprehensive survey). The first approach has been to formulate a set of intuitively appealing criteria as axioms, and then characterize the *division rules* according to the axioms they satisfy. The second approach poses the division problem as either an axiomatic bargaining problem, or

[1] See Moulin (2003) for a discussion of these issues.

[2] This absence might be due to several reasons. For example, people probably realize that marriage might end up in divorce, but find it unromantic to write a pre-nuptial agreement.

a coalitional bargaining problem, and characterizes the corresponding bargaining solutions. It has also been shown that most of the division rules correspond to some axiomatic/cooperative bargaining solution. A sound non-cooperative justification for the division rules has not been provided.

Consider the problem of ex-post division of a bankrupt estate e between two anonymous claimants who had independently contributed c_l and c_h. All the division rules prescribe dividing the estate *efficiently* while satisfying *claims boundedness*. Efficiency requires that the payoffs to the agents must sum up to the amount to be divided. Claims boundedness requires that no claimant should obtain more than her initial contribution for any $e \leq (c_l + c_h)$. The division rules differ with respect to the *additional* criteria that should be satisfied. The additional criterion satisfied by the Constrained Equal Awards (CEA) rule is that the variance of agents' payoffs should be minimum. The CEA rule is a natural choice in scheduling problems (Shenker, 1995). The additional criterion satisfied by the Proportional $(PROP)$ rule is homogeneity, which requires that the division should be independent of the units of measurement. The Proportional rule is widely used in both formal and informal environments (Knight, 1992, Ellickson, 1991).

We are interested in providing a non-cooperative justification for the division rules. Suppose, two agents after having independently contributed $(c_l = 0.3, c_h = 0.7)$ find themselves in a bankruptcy like situation, and bargain in the framework of the *Nash demand game* in order to divide the estate $e = 0.5$. Any pair of demands that sum to $e = 0.5$ will be a Nash equilibrium. However, we believe the equilibrium demand pair $(d_l = 0.45, d_h = 0.05)$ is an unlikely outcome since the low claimant obtains more than her initial contribution. We, therefore, add to the rules of Nash demand game the axiom of claims boundedness which prescribes that the payoff to an agent should never exceed her initial contribution for any $e \leq (c_l + c_h)$. This modified demand game will be referred to as the *augmented Nash demand game*. The demand vector $(d_l = 0.45, d_h = 0.05)$ will lead to payoffs $(0.3, 0.05)$ in the one-shot augmented Nash demand game.

The equilibrium strategies in the augmented Nash demand game are the same as in the usual Nash demand game. More importantly, outcomes prescribed by all the division rules are strict Nash equilibria in the one-shot augmented Nash demand game. The paper proves that if we embed the augmented Nash demand game in Young's (1993b) evolutionary bargaining model then the unique long run stochastically stable outcome is the division prescribed by the CEA rule.

The demand (d_i) by an agent can also be interpreted as specifying the division of the estate, $(x_i, x_j) = (d_i, e - d_i)$. This view provides the motivation for considering the *contracting game* (Young, 1998a) wherein the demand of an agent is interpreted as a proposal for the efficient division of the estate. Agents obtain strictly positive payoffs only when they propose the same efficient division (or, the sum of demands must be exactly e). Augmenting the contracting game by adding the axiom of claims boundedness leads to the emergence of the division suggested by the Truncated Claims Proportional (TCP) rule as the unique long run outcome.

Consider the case with $(c_l = 0.3, c_h = 0.7, e = 0.5)$. The maximum feasible payoff to the high claimant is $min(c_h, e) = min(0.7, 0.5) = 0.5$, and she will end up losing at least $[c_h - min(c_h, e)] = 0.2$. This part of the initial contribution is sunk as it is beyond recovery. The maximum payoff feasible for agent i, $min(c_i, e)$, is defined as the truncated claim of agent i. The long run outcome turns out to be the division of e in proportion to the truncated claims when bargaining occurs in the framework of the *augmented contracting game*. This division coincides with the Kalai-Smorodinsky (1975) bargaining solution. The TCP rule and the Proportional rule suggest the same division only when the leftover estate is at least equal to the highest contribution.

The paper is organized as follows. Section 2 describes the set up of the model. Section 3 contains the main results of the paper and briefly discusses the robustness of the results. Section 4 concludes. The appendix contains proofs of the propositions.

2. The Model

The economy is characterized by the tuple (c_l, c_h, e), where $c_h \geq c_l \geq \delta$, and $2\delta \leq e \leq (c_l + c_h) = 1$. The parameter δ reflects the least count of the monetary scale used in the economy. The economy consists of two distinct populations (low claimants and high claimants) of equal size N. The agents are assumed to have independently contributed c_l and c_h in the past, and come together in the form of a bargaining pair (consisting of one L-claimant and one H-claimant) only after the realization of bankruptcy to decide upon the division of the remaining estate e. Note that, the decision of agents regarding whether to contribute, and if so how much, is not being modeled explicitly. (Suppose the agents have to decide how much to invest in a risky project and how to divide the surplus. From an ex-ante perspective, there will still exist multiple ways of dividing the risky surplus that will make it individually rational for the agents to invest). Thus, in a particular economy, c_l, c_h, and e take the same[3] numerical values across all pairs for all time periods. In period t the demand of agent i is: the inertial demand $d_i(t-1)$ with probability $(1-\alpha)$, a best response to the average demand of agents in the other population during the previous period $(\overline{d}_j(t-1))$ with probability $(1-\lambda)\alpha$, or a random demand[4] from the feasible set of demands with probability $\lambda\alpha$.

Let the state of the economy at the end of period t be defined as $s_t = (n_t^l, n_t^h)$, where n_t^j is a K dimensional vector representing the number of agents in population $j \in (L, H)$ who played the pure strategy $k \in \{1, \ldots, K\}$ (assuming that $e = (K+1)\delta$) during period t. This dynamic specification can be concisely represented as a Markov chain M_λ on the finite state space S consisting of all pairs

[3]If we allow the values of c_l, c_h, and e to vary then we will need a satisfactory model of learning when the strategy space changes with time.

[4]This specification is similar to the random best response dynamics of Binmore, Samuelson, and Young (BSY, 2003). The source of randomness does not matter. For example, it can be an error, or an experiment motivated by the desire to obtain a greater share for oneself, or by a concern for the other agent.

$s = (n^l, n^h) \in R^K \times R^K$, with $\sum n_k^l = \sum n_k^h = N$. Every state is accessible from every other state in a finite number of periods because agents state random demands with strictly positive probability during each period. M_λ is therefore irreducible. It is aperiodic as well because there does not exist any state to which the process will continually return with a fixed time period. Irreducibility implies that the process can potentially escape even a Nash equilibrium state as with non-zero probability of random play Nash equilibria cease to be the absorbing states. Irreducibility, together with aperiodicity, implies that the stationary probability distribution over states will be unique and independent of the initial state. Let $v_\lambda(s_t|s_0)$ be the relative frequency of the occurrence of state s till time t, given the initial state is s_0. Then,

$$\lim_{t\to\infty} v_\lambda(s_t|s_0) = \mu_\lambda(s). \tag{1}$$

2.1 *The Underlying Bargaining Games*

The two agents are assumed to have independently contributed c_l and c_h, and are required to bargain over the leftover estate e amongst themselves. The role of the implicit neutral arbiter is to enforce the rules of bargaining. We consider two one-shot bargaining games- the augmented contracting game (cg), and the augmented Nash demand game (dg). For a given, (c_l, c_h, e), let $G(cg)$, and $G(dg)$, represent these two games, respectively. Agents state one and only one demand d_i from the discrete and finite set $\{\delta, 2\delta, \ldots, e-\delta\}$ during the bargaining. The rules of bargaining determine the payoffs (x_l, x_h) resulting from the demands (d_l, d_h). The rules for the two games are summarized in Table 1.

Let the pair of demands $d = (d_l, d_h) \in D$, with $d_l, d_h \in \{\delta, \ldots, (e-\delta)\}$, represent a pure strategy vector. For a given (c_l, c_h, e), we have the following lemma. The proof is straightforward, and hence omitted.

Table 1: RULES OF BARGAINING

Game	Payoffs resulting from (d_l, d_h)
$G(cg)$	$(min(c_l, d_l), min(c_h, d_h))$ iff $d_l + d_h = e$
$G(dg)$	$(min(c_l, d_l), min(c_h, d_h))$ iff $d_l + d_h \leq e$

Lemma 1. $D^*(cg) = D^*(dg)$. The demand pair $d^* = (d_l^*, d_h^*)$ is a pure strategy Nash equilibrium if and only if $d_l^* + d_h^* = e$.

Outcomes prescribed by all the existing division rules (CEA, $PROP$, $TCPR$, etc.) are strict Nash equilibria in each of the one shot games. Our aim is to find whether the evolutionary process converges to a unique outcome in the long run. And if so, does the resulting division correspond to the payoffs prescribed by a particular division rule. The most promising way to address the issue of equilibrium selection in the presence of multiple strict Nash equilibria is stochastic stability which tries to understand the behavior of a dynamic process in the presence of persistent randomness. Other approaches to analyze stability of an equilibrium involve perturbing the system only after it has settled into a steady state. Stochastic stability is concerned with finding which of the several steady states of the unperturbed process are stable (in a sense described below) in the presence of continuous perturbations arising from the possibility of random play by agents.

2.2 Stochastic Stability

The one-shot bargaining games have multiple pure strategy Nash equilibria.

Considerations of stochastic stability allow selection even among multiple strict Nash equilibria. Stochastic stability relates to the limit of the stationary distribution of the Markov process as the probability of random play goes to zero. The state s^* is stochastically stable if

$$\lim_{\lambda \to 0} \mu_\lambda(s^*) > 0. \tag{2}$$

The stochastically stable state is the one most likely to be observed in the long run as the probability of random play by agents tends to zero. Intuitively, such a state is easy to reach but difficult to escape via random play. Only the pure strategy Nash equilibria *can* be the absorbing states of M_0. Hence, we do not need to consider the transition matrices specifying the probabilities of transition from every state to every other state. Stochastic stability calculations involve checking for the ease of transition only among the equilibrium states. We can use the mutation counting technique[5] (BSY 2003) to find the set of stochastically stable states of M_λ. Next, we provide some useful definitions from graph theory, and briefly describe how to use the mutation counting technique to identify the stochastically stable state(s) (see Young, 1993a, 1998a, 1998b, and BSY, 2003, for further details.).

A graph consists of two types of elements: nodes and edges. An edge connects a pair of nodes. A graph in which the edges have a sense of direction are called directed graphs. A node is reachable from some other node in the graph if there is a directed path that starts at the latter and ends at the former. The graph is connected if it is possible to establish a path from any node to any other node in the graph. A tree is a connected graph with no cycles. In rooted trees the number of edges is one less than the number of nodes such that each edge is directed towards the root node, and from every node there is one and only one directed

[5]The payoff structure of the games suggests that they satisfy the no cycling condition of Young (1993a) and the marginal bandwagon property of Kandori and Rob (1998).

path to the root node. However, there can be several trees rooted at the same node.

Consider the complete set of directed graphs constructed by using each pure strategy Nash equilibrium of the underlying bargaining game as a root. The resistance (or, cost) of the directed edge joining equilibrium k_1 to equilibrium k_2 is the minimum number of experimenting agents required to move the process from K_1 to k_2. Consider any one of the trees rooted at, say, the equilibrium $(d_l^* = k\delta, d_h^* = (K-k)\delta)$. The resistance of a tree rooted at this k^{th} equilibrium is defined as the sum of the resistances of the edges along its path. The resistance of each tree, rooted at each equilibrium, can be calculated in a similar manner. The root of the tree with the minimal total resistance is the stochastically stable equilibrium.

3. The long run equilibrium

The pure strategy Nash equilibria of the one shot bargaining games have been described in the previous section. Now, we attempt to identify which one of these emerges as the long run outcome using the mutation counting technique. Recall that the process can escape a Nash equilibrium only because of random play and stochastic stability is concerned with the long run outcome when the probability of random play tends to zero. With vanishingly small probability of random play, what ultimately matters is which equilibrium is relatively easy to reach but difficult to escape via random play.

3.1 Contracting game with claims boundedness $G(cg)$

Suppose, the process is currently in the equilibrium (d_l, d_h). (Henceforth, we omit the asterisk sign over the equilibrium demands). This will often be referred to as the equilibrium at d_l. Since agents experiment with non best response strategies, the process can over time move from any equilibrium to any other equilibrium. Define D_l^+ as the set of equilibrium demands by the low claimant higher than d_l, and D_l^- as the set of equilibrium demands by the low claimant lower than d_l. We can analogously define D_h^+, and D_h^-. Let d_l^+ be a representative element of D_l^+. The least costly transition *out* of the equilibrium at d_l is to that equilibrium which requires least number of agents to experiment. It can either be in D_l^+, or in D_l^-. We will separately figure out the most easily accessible equilibrium to the right of a given d_l lying in D_l^+, and to the left of d_l lying in D_l^-. The easier of these two will in turn be termed as the least costly transition out of the equilibrium at d_l. The relevant 2×2 games that need to be considered when $G(cg)$ is the underlying bargaining game are shown in Figure 1.

	d_l^-	d_l
d_h^+	$min(d_h^+, c_h)$, $min(d_l^-, c_l)$	0 , 0
d_h	0 , 0	$min(d_h, c_h)$, $min(d_l, c_l)$

	d_l	d_l^+
d_h	$min(d_h, c_h)$, $min(d_l, c_l)$	0 , 0
d_h^-	0 , 0	$min(d_h^-, c_h)$, $min(d_l^+, c_l)$

Fig. 1. The representative 2×2 games in $G(cg)$.

The basic intuition behind all the calculations in the paper is as follows. (i) Suppose the economy is in the equilibrium (d_l, d_h) at time t. (ii) There is positive probability that a sufficiently high fraction of L-agents happen to randomly state the demand $d_l^+ > d_l$ at time $(t+1)$, the remaining fraction of L-agents still demand d_l, and all H-agents also behave inertially such that $d_h(t+1) = d_h$. (iii) With a positive probability all the L-agents behave inertially during $(t+2)$ such that $d_l(t+2) = d_l(t+1)$, while all the H-agents best respond to the average demand of L-agents during the previous period ($\bar{d}_l(t+1)$) and state the demand $d_h(t+2) = d_h^-$. (iv) Hence, there is a positive probability that the process ends up in the equilibrium (d_l^+, d_h^-) at time $(t+3)$. This will require that the L-agents best respond to $\overline{d}_h(t+2) = d_h^-$ by demanding $d_l(t+3) = d_l^+$, but all the H-agents behave inertially and demand $d_h(t+3) = d_h(t+2) = d_h^-$.

Two things are worth noting. First, random play is required only to initiate the transition $[d_l \rightarrow d_l^+]$. Once a sufficiently high fraction of agents in a population experiment, the process *can* move out of the current equilibrium and end up in some other equilibrium without any further random play. Second, the minimum fraction of agents that must experiment in order to ensure that there is a strictly positive probability of transition depends both on the origin (d_l) and the destination (d_l^+). In $G(cg)$, the smaller is d_l^+ the lower will be the fraction of L-agents required to experiment with d_l^+ to ensure that the transition $[d_l \rightarrow d_l^+]$ happens with strictly positive probability. In general, the least costly transition out of the current equilibrium in $G(cg)$ can be initiated by some agents in either population randomly stating -

$[1]$ **Higher experimental demands**: Suppose the agents in population i experiment with a demand higher than their current equilibrium demand, i.e., $d_i^+ > d_i$. If d_i^+ is established as an equilibrium then j-agents can not obtain more than what they were getting in the equilibrium at (d_i, d_j). The least costly transition involves i-agents experimenting with their minimal higher demand of $(min(d_i^+))$. Though it leads to a decrease in the equilibrium payoff of j-agents, the decrease will be the minimum possible. This type of least costly transition will be referred to as the *local* transition.

$[2]$ **Lower experimental demands**: Suppose the agents in population i experiment with a demand lower than their current equilibrium demand, i.e., $d_i^- < d_i$. The least costly transition will involve these agents experimenting with their lowest possible equilibrium demand $(min(d_i^-) = min(d_i))$ such that the corresponding equilibrium payoff to the j-agents is the maximal increased payoff of

$Max[min(d_j^+, c_j)] = Max[min(d_j, c_j)]$. This type of least costly transition will be referred to as the *extreme* transition.

Transitions towards the right (or the left) of the existing equilibrium can be initiated by agents in either population. In $G(cg)$, for an equilibrium at any $d_l \in D_l$, the least costly transition towards the right into the corresponding D_l^+ turns out to be the transition to the *extreme* right involving the H-agents experimenting with their lowest demand of $min(d_h^-) = min(d_h)$. Define $r^+(d_l)$, the resistance (or, the cost) of this least costly transition $[d_l \rightarrow max(d_l)]$ as the minimum fraction of H-agents that must experiment to accomplish this transition with positive probability. In $G(cb)$,

$$r_{out}^+(d_l) = \frac{min(d_l, c_l)}{min(d_l, c_l) + Max[min(d_l, c_l)]}, \quad \forall\, d_l \in D_l. \tag{3}$$

Similarly, the least costly transition out of the equilibrium at d_l towards the left into D_l^- happens to be the transition to the *extreme* left $[min(d_l) \leftarrow d_l]$ and involves the L-agents demanding $min(d_l)$. The minimum fraction of L-agents that must experiment to accomplish this transition is

$$r_{out}^-(d_l) = \frac{min(d_h, c_h)}{min(d_h, c_h) + Max[min(d_h, c_h)]}, \quad \forall\, d_l \in D_l. \tag{4}$$

The least costly transition out of the equilibrium at any given d_l will be the one which has the lower cost among these two extreme transitions. Given an existing equilibrium at d_l, the least costly transition will be to the equilibrium on extreme right if

$$r_{out}^+(d_l) \leq r_{out}^-(d_l). \tag{5}$$

Or, if

$$\frac{min(d_l, c_l)}{Max[min(d_l, c_l)]} \leq \frac{min(d_h, c_h)}{Max[min(d_h, c_h)]}. \tag{6}$$

Note that $min(d_l, c_l)$ is the payoff of the L- claimant in the Nash equilibrium at d_l. $Max[min(d_l, c_l)]$ and $Max[min(d_h, c_h)]$ are the highest possible Nash equilibrium payoffs to the low claimant and high claimant respectively. Also, $r^+_{out}(d_l)$ is (weakly) increasing in d_l and $r^-_{out}(d_l)$ is (weakly) decreasing in d_l. This means that the least costly transition towards the right (left) from the existing equilibrium at d_l becomes more costly if d_l is high (low). Thus, there exists[6] a $d_l^*(cg, cb)$ such that the cost of the least costly transition to left of $d_l^*(cg, cb)$ is the same as the cost of the least costly transition to right of $d_l^*(cg, cb)$. The equilibrium at $d_l^*(cg, cb)$ is the most difficult to escape as it

$$\text{Maximizes} \quad Minimum\,(r^+_{out}(d_l),\ r^-_{out}(d_l)), \qquad \forall\, d_l \in D_l. \tag{7}$$

Or,

$$\text{Maximizes} \quad Minimum\,[\frac{min(d_l^*(cg, cb), c_l)}{Max[min(d_l, c_l)]}, \frac{min(d_h^*(cg, cb), c_h)}{Max[min(d_h, c_h)]}], \qquad \forall\, d_l \in D_l. \tag{8}$$

The two terms inside the brackets can be interpreted as welfare indices (W_l, W_h) of the two claimants in the equilibrium at $d_l^*(cg, cb)$. If we think of the welfare index of agent i as the ratio of her payoff in an equilibrium to the maximum possible equilibrium payoff (Young, 1998a), then the equilibrium at $d_l^*(cg, cb)$, by virtue of being the most difficult equilibrium to escape,

$$\text{Maximizes} \quad Minimum\,(W_l(d_l),\ W_h(d_l)), \qquad \forall\, d_l \in D_l. \tag{9}$$

[6] The discreteness of the strategy space implies that this may not be so. However, it does not change any of the results if δ is small.

Proposition 1. For a given (c_l, c_h, e), the payoffs to the agents in the long run stochastically stable outcome of the dynamic process with $G(cg)$ as the underlying bargaining game coincide with those suggested by the truncated claims proportional rule. Therefore,

$$\frac{x_l^{ss}(cg, cb)}{x_h^{ss}(cg, cb))} = \frac{d_l^{ss}(cg, cb)}{d_h^{ss}(cg, cb))} = \frac{Max[min(d_l, c_l)]}{Max[min(d_h, c_h)]}. \tag{10}$$

Proof. In Appendix $A.1$.

The maximum equilibrium payoffs to the agents are

$$(Max[min(d_l, c_l)], Max[min(d_h, c_h)]) = \begin{cases} (e-\delta,\ e-\delta) & \text{if } \ e \leq c_l. \\ (c_l,\ e-\delta) & \text{if } \ c_l < e \leq c_l. \\ (c_l,\ c_h) & \text{if } \ c_h < e. \end{cases} \tag{11}$$

The payoffs to the agents in the long run equilibrium (for $\delta \to 0$) are

$$(x_l^{ss}, x_h^{ss}) = (d_l^{ss}, d_h^{ss}) = \begin{cases} ([\frac{e}{e+e}]e,\ [\frac{e}{e+e}]e) & \text{if } \ e \leq c_l. \\ ([\frac{c_l}{c_l+e}]e,\ [\frac{e}{c_l+e}]e) & \text{if } \ c_l < e \leq c_l. \\ ([\frac{c_l}{c_l+c_h}]e,\ [\frac{c_h}{c_l+c_h}]e) & \text{if } \ c_h < e. \end{cases} \tag{12}$$

The truncated claims proportional rule first defines the truncated claim of each agent as $min(c_i, e)$; and then divides the estate proportionally to the truncated claims. It exactly mirrors the Kalai-Smorodinsky (1975) solution, and Gauthier's (1986) principle of mini-max relative concession with $(min(c_l, e), min(c_h, e))$ as the initial bargaining position. The proportional rule would suggest dividing the estate in proportion to the initial contributions irrespective of the size of the leftover estate. The sunk claims can not be accounted for during the ex-post bargaining in our framework, and thus truncated proportionality emerges instead of exact proportionality. A similar result is obtained when the claims problem is posed as a coalitional bargaining problem. While defining the ex-post worth of coalitions, it is the truncated claims that matter and not the original contributions. This is the reason no existing coalitional game solution concept prescribes an outcome that corresponds to the proportional division rule.

3.2 Nash demand game with claims boundedness

The pure strategy strict Nash equilibria of the Nash demand game with claims boundedness are exactly similar to those of the contracting game with claims boundedness. However, the two payoff matrices differ as in the demand game agents obtain payoffs even if the sum of demands is less than the estate (see Figure 2). This is a crucial difference as the least costly transition from the equilibrium at d_l in $G(dg)$ will differ from that in $G(cg)$. The underlying logic for identifying the least costly transitions in $G(dg)$ is exactly similar to that in $G(cg)$.

	d_l^-	d_l
d_h^+	$min(d_h^+, c_h)$, $min(d_l^-, c_l)$	0 , 0
d_h	$min(d_h, c_h)$, $min(d_l^-, c_l)$	$min(d_h, c_h)$, $min(d_l, c_l)$

	d_l	d_l^+
d_h	$min(d_h, c_h)$, $min(d_l, c_l)$	0 , 0
d_h^-	$min(d_h^-, c_h)$, $min(d_l, c_l)$	$min(d_h^-, c_h)$, $min(d_l^+, c_l)$

Fig. 2. The representative 2×2 games in $G(dg)$.

However, the exact fraction of agents required to accomplish a transition will be different.

Let $f_l[d_l \rightarrow d_l^+]$ denote the minimum fraction of L-agents that must experiment with the higher demand of d_l^+ in order that the process transits from the equilibrium at d_l to the equilibrium at d_l^+ with a positive probability. The second 2×2 game in Figure 2 gives

$$f_l[d_l \rightarrow d_l^+] \;=\; \frac{min(d_h, c_h) - min(d_h^-, c_h)}{min(d_h, c_h)}, \quad \forall\, d_l \in D_l. \tag{13}$$

Similarly, the minimum fraction of H-agents that must experiment with the lower demand of d_h^- in order that the process transits from the equilibrium at d_l to d_l^+ with a positive probability is

$$f_h[d_l \rightarrow d_l^+] \;=\; \frac{min(d_l, c_l)}{min(d_l^+, c_l)}, \quad \forall\, d_l \in D_l. \tag{14}$$

The value of $f_l[d_l \rightarrow d_l^+]$ is minimized when $min(d_h^-, c_h)$ is maximum. This in turn implies that the least costly transition initiated by L-claimants towards $d_l^+ > d_l$ is the local transition $[d_l \rightarrow min(d_l^+)]$. Similarly, the least costly transition initiated by H-claimants towards $d_l^+ > d_l$ is the extreme transition $[d_l \rightarrow max(d_l)]$.

The overall least costly transition towards the right happens to be the *local* transition initiated by experiments of L-agents. For this result to be true it has to be proved that the local transition initiated by experiments of L-claimants requires less number of experimenting agents than the extreme transition initiated by the experiments of H-claimants. Formally, we require

$$f_h[d_l \rightarrow max(d_l)] \geq f_l[d_l \rightarrow min(d_l^+)], \quad \forall\, d_l \in D_l. \tag{15}$$

Or,

$$min(d_l, c_l).min(d_h, c_h) \geq Max[min(d_l, c_l)].(min(d_h, c_h) - Max[min(d_h^-, c_h)]), \quad (16)$$

must hold for all $d_l \in D_l$. It can be easily verified that the above inequality indeed holds true (calculations are provided in the appendix.). The term on the left can be thought of as the *constrained Nash product* (the product of *truncated* payoffs) at the current equilibrium (d_l, d_h). The inequality essentially means that the least costly transition towards the right into D_l^+ from the equilibrium at any $d_l \in D_l$ is to that equilibrium in D_l^+ which has the highest constrained Nash product.

The procedure for calculating the most easily accessible equilibrium towards the left of the current equilibrium at d_l is the same. The least costly transition from any equilibrium d_l towards the left into D_l^- turns out be the local transition initiated by the experiments of H-claimants. Finally, the most easily accessible equilibrium out of these two local transitions from the equilibrium at d_l is the one with a higher constrained Nash product. This can be interpreted to imply that the evolutionary process has a tendency to move towards the pure strategy Nash equilibrium with the highest constrained Nash product. If we think of the payoffs as a measure of welfare, then the process has a tendency to move towards the Nash equilibrium which

$$\text{Maximizes} \quad [W_l(d_l) \cdot W_h(d_l)], \qquad \forall \, d_l \in D_l. \quad (17)$$

It is clear that the division of e in this welfare maximizing Nash equilibrium is the efficient division that minimizes the difference between the payoffs of the two agents, and satisfies claims boundedness.

Proposition 2. For a given (c_l, c_h, e) the payoffs to the agents in the long run stochastically stable outcome of the dynamic process with $G(dg)$ as the underlying bargaining game coincide with those suggested by the constrained equal awards rule. Therefore,

$$(x_l^{ss}, x_h^{ss}) \;=\; (d_l^{ss}, d_h^{ss}) \;=\; \begin{cases} (\frac{1}{2}e, \frac{1}{2}e) & \text{if } \; e \leq 2c_l. \\ (c_l, e - c_l) & \text{if } \; e > 2c_l. \end{cases} \tag{18}$$

Proof. In Appendix $A.2$.

Reconsider the example with $(c_l, c_h) = (0.3, 0.7)$. If $e = 0.5 \leq 2c_l = 0.6$, the CEA rule will give $0.5e = 0.25$, to both agents. Equal division maximizes the constrained Nash product among all pure strategy Nash equilibria if $e \leq 2c_l$. However, if $e > 2c_l$ (say, 0.9), then CEA prescribes $c_l = 0.3$ for the low claimant and $(e - c_l) = 0.6$ for the high claimant. The division $(c_l, e - c_l)$ maximizes the constrained Nash product among all pure strategy Nash equilibria if $e > 2c_l$.

3.3 The games without claims boundedness

The pure strategy Nash equilibrium strategies in the games without claims boundedness will be the same as in the games with claims boundedness. However, we know that the payoffs resulting from the equilibrium demands will equal the demands, in the usual Nash demand game and the usual contracting game.

Let us first consider the contracting game without claims boundedness. The least costly transitions will be determined exactly in the same way as in $G(cg)$. But, the maximum equilibrium payoff to each agent will now be $(e - \delta)$ for all $e \leq 1$.

Corollary 1. For a given (c_l, c_h, e), equal division is the long run stochastically stable outcome of the evolutionary process with $G(cg)$ as the underlying bargaining game.

Proof. It is clear that

$$\frac{x_l^{ss}(cg)}{x_h^{ss}(cg))} = \frac{d_l^{ss}(cg)}{d_h^{ss}(cg))} = \frac{(e-\delta)}{(e-\delta)} = 1. \tag{19}$$

The rules of the contracting game without claims boundedness do not account for the initial contributions in any way. Thus, this result is identical to the one obtained by Young (1998a) and BSY (2003) in bargaining over one unit of exogenously given surplus. Similarly, in the demand game without claims boundedness, equal division will maximize the product of payoffs, and thus be the long run outcome (Young, 1993a).

3.4 Robustness

We have assumed that random play involves agents playing any strategy randomly from the feasible set with uniform probability. (See Bergin and Lipman (1996) for the implications of non-best response play with state dependent probability distribution over the set of feasible strategies). Suppose agents commit these errors/experiments rationally (Naidu and Bowles, 2006). Then, if the economy is currently in the equilibrium at d_l, the low agent will never experiment with a demand less than d_l, and the high claimant will never experiment with a demand less than $d_h = (e - d_l)$. Transitions can then be initiated only by agents asking for more than what they are getting in the existing equilibrium. This in turn implies that the least costly transitions will always be the *local* transitions as the *extreme* transitions are initiated by agents demanding less than what they are getting in the existing equilibrium. Thus, even in the contracting game with claims boundedness ($G(cg)$) the long run outcome will be the division prescribed by the constrained equal awards rule. Similarly, if we had employed the continuous best response dynamics (BSY, 2003)) instead of the random best response dynamics, the long run outcome in $G(cg)$ would again be given by the CEA rule. The CEA outcome thus seems more robust to alternative specifications than the TCP outcome. In fact, Gaechter and Riedl (2006) provide experimental evidence that when agents bargain amongst themselves over the division of a surplus that falls short of the sum of their initial claims, the outcome is closest to the CEA rule.

4. Conclusion

The problem of dividing scarce resources among multiple agents with competing claims arises in several contexts. The existing literature provides several division rules from the perspective of a neutral arbiter. This paper explores what allocation of a divisible scarce resource will emerge in the long run if agents bargain non-cooperatively amongst themselves. The division suggested by the truncated claims proportional rule is the long run outcome of the evolutionary process when the bargaining between the agents occurs in the framework of the contracting game, augmented with the axiom of claims boundedness. If the augmented Nash demand game is used as the underlying bargaining game, then the long run outcome is the division prescribed by the constrained equal awards rule.

Proportional division is probably the most appealing solution. The most important property of the proportional rule is that a coalition of agents can not benefit from strategically transferring a part of their claims to one another. The truncated claims proportional rule is quite similar and suggests the same division as the proportional rule when the resource is greater than the initial claims of both agents. Binmore (2005) rationalizes proportionality as the outcome of long run evolution of empathetic preferences which involves interpersonal comparisons. The approach used in the present paper does not require agents to make any interpersonal comparisons.

The behavioral specification of agents in this paper is not only standard in the stochastic stability literature, but also reasonable. A drawback of our set up is the assumption of fixed values of c_l, c_h, and e for all pairs, and time periods. Suppose, otherwise, that $(c_l, c_h) = (0.4, 0.6)$ for all pairs at all times, but the realization of e can be 0.5, or 0.9. Should the model allow the two agents in a pair that are bargaining over $e = 0.5$ during the current period to draw inferences from the past play in cases with $e = 0.9$? If yes, then we need to be able to model learning when the strategy space changes over time.[7] If not, and players learn only from the plays in the previous period that had $e = 0.5$, then allowing for two values of e is redundant. The existing literature has implicitly taken the latter route. Ellingsen and Robles (2002), and Troeger (2002) develop evolutionary models in which two agents bargain over a surplus that is created by one agent's investment, and show that evolution eliminates the hold up problem. But, they assume that agents state their demands by consulting the distribution of past demands of opponents only in cases that had the same amount of surplus. This question of learning from different situations does not even arise in Young (1993b) and BSY (2003) as the bargaining always takes place over one unit of surplus.

We have also assumed the initial contributions to be exogenously given. Ellingsen and Robles (2002) show that while efficient investment can arise in the long run under several rules of bargaining, the division of the surplus is sensitive to the rules. This suggests that an attempt rationalize a particular division of the surplus as the long run outcome should primarily focus on the rules of bargaining.

Finally, this paper deals with the simplest possible allocation problem with well defined claims over a divisible and homogenous resource. Elster (1991) provides a fascinating discussion of the various ways in which institutions allocate scarce resources depending upon whether the resource is divisible, and all the units are homogenous, or not. It remains to be seen whether evolutionary analysis can provide any new insights in such cases.

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[7] We could restrict the strategy space to the $[0, 1]$ interval by assuming that agents demand fractions of the estate during bargaining. But, then we need to explain why so.

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Appendix A.

A.1. Proof of Proposition 1

The second 2×2 game in Figure 1 helps us calculate the fraction, $f_l[d_l \rightarrow d_l^+]$, of agents in the L-population that should experiment with the higher demand of d_l^+ such that the best response for agents in the H-population is to demand d_h^-. In $G(cg)$,

$$f_l[d_l \rightarrow d_l^+] \;=\; \frac{min(d_h, c_h)}{min(d_h, c_h) + min(d_h^-, c_h)}, \quad \forall\, d_l \in D_l. \tag{20}$$

The least costly transition from d_l into D_l^+ initiated by experiments of L- agents asking for more will be the *local* transition. The associated cost will be

$$f_l[d_l \rightarrow min(d_l^+)] \;=\; \frac{min(d_h, c_h)}{min(d_h, c_h) + Max[min(d_h^-, c_h)]}, \quad \forall\, d_l \in D_l. \tag{21}$$

The least costly transition from d_l into D_l^+ initiated by experiments of H- agents asking for less will be the *extreme* transition. The associated cost will be

$$f_h[d_l \to max(d_l)] \;=\; \frac{min(d_l, c_l)}{min(d_l, c_l) + Max[min(d_l, c_l)]}, \quad \forall\, d_l \in D_l. \tag{22}$$

Given the equilibrium at d_l the extreme transition initiated by the experiments of high claimants requires lesser number of agents to experiment. Note that $f_l[d_l \to min(d_l^+)] \geq f_h[d_l \to max(d_l)]$ since

$$\frac{min(d_h, c_h)}{Max[min(d_h^-, c_h)]} \geq 1 \geq \frac{min(d_l, c_l)}{Max[min(d_l, c_l)]}, \quad \forall\, d_l \in D_l. \tag{23}$$

Thus, the most easily accessible equilibrium out of a pure strategy Nash equilibrium at any d_l towards the right in the corresponding D_l^+ is the extreme transition initiated by the H-claimants. We defined $r^+(d_l)$, the resistance of the least costly transition from the equilibrium at d_l into D_l^+, as the minimum fraction of agents that must experiment to accomplish this transition. In $G(cb, cg)$,

$$r_{out}^+(d_l) \;=\; \frac{min(d_l, c_l)}{min(d_l, c_l) + Max[min(d_l, c_l)]}, \quad \forall\, d_l \in D_l. \tag{24}$$

Similar calculations for the transitions towards left of d_l into D_l^- show the the least costly transition is to the extreme left initiated by L-agents asking for the least possible equilibrium payoff. It's resistance is

$$r_{out}^-(d_l) \;=\; \frac{min(d_h, c_h)}{min(d_h, c_h) + Max[min(d_h, c_h)]}, \quad \forall\, d_l \in D_l. \tag{25}$$

In $G(cg)$, the least costly transition from a given equilibrium at $d_l \in D_l$ is the extreme transition into D_l^+ (D_l^-) if $d_l < \ (>) d_l^{ss}$. Moreover, the equilibrium at d_l^{ss} is the most difficult to escape.

Proceeding in a similar manner we now try to find what is the easiest way to get into the equilibrium at any $d_l \in D_l$. We have

$$f_l[d_l \leftarrow d_l^+] \;=\; \frac{min(d_h^-, c_h)}{min(d_h^-, c_h) + min(d_h, c_h)}, \quad \forall\, d_l \in D_l. \tag{26}$$

$f_l[d_l \leftarrow d_l^+]$ is minimum if $min(d_h^-, c_h)$ is as small as possible. Similarly,

$$f_h[d_l \leftarrow d_l^+] \;=\; \frac{min(d_l^+, c_l)}{min(d_l^+, c_l) + min(d_l, c_l)}, \quad \forall\, d_l \in D_l. \tag{27}$$

gets minimized when $min(d_l^+, c_l)$ is minimum. The minimum value of $f_l[d_l \leftarrow d_l^+]$ is less than the minimum value of $f_h[d_l \leftarrow d_l^+]$ as

$$\frac{min(d_h, c_h)}{Min[min(d_h^-, c_h)]} \geq 1 \geq \frac{min(d_l, c_l)}{Min[min(d_l^+, c_l)]}, \quad \forall\, d_l \in D_l. \tag{28}$$

Hence, the least costly transition into the equilibrium at d_l from any equilibrium in D_l^+ is the transition from the extreme right. It can similarly be shown that the least costly transition into the equilibrium at d_l from any equilibrium in D_l^- is the transition from the extreme left. The resistance of getting into the equilibrium at d_l from the right and from the left, respectively, is

$$r_{in}^+(d_l) \;=\; \frac{Min[min(d_h^-, c_h)]}{Min[min(d_h^-, c_h)] + min(d_h, c_h)} \;=\; \begin{cases} \frac{\delta}{\delta + c_h} & \text{if } \; d_l \leq e - c_h. \\ \frac{\delta}{\delta + e - d_l} & \text{if } \; d_l > e - c_h. \end{cases} \tag{29}$$

$$r_{in}^-(d_l) \;=\; \frac{Min[min(d_l^-, c_l)]}{Min[min(d_l^-, c_l)] + min(d_l, c_l)} \;=\; \begin{cases} \frac{\delta}{\delta + d_l} & \text{if } \; d_l \leq c_l. \\ \frac{\delta}{\delta + c_l} & \text{if } \; d_l > c_l. \end{cases} \tag{30}$$

The resistance of getting into any $d_l \in D_l$ will be given by the minimum of $r_{in}^+(d_l)$ and $r_{in}^-(d_l)$. However, for $\delta \to 0$ these resistances tend to zero.

Consider now the tree with minimum total resistance rooted at, say, $d_l^c < d_l^{ss}$. It will be obtained by: (1) Connecting all $d_l \neq d_l^c \in (d_l^{min}, d_l^{ss})$ to d_l^{max}. (2) Connecting all $d_l \in [d_l^{ss}, d_l^{max})$ to d_l^{min}. (3) Connecting d_l^{min} to d_l^{max} to d_l^c, or d_l^{max} to d_l^{min} to d_l^c depending upon which of these two transition sets has the lower resistance. The tree with minimum total resistance rooted at, any $d_l^c \geq d_l^{ss}$ can be analogously obtained. We will now show that the resistance of the tree rooted at any $d_l^c \neq d_l^{ss}$ is greater than the resistance of the tree rooted at d_l^{ss}.

$$R(d_l^c) \ = \ \sum_{d_l \neq d_l^{min}, d_l^c, d_l^{max}} min[r_{out}(d_l \rightarrow d_l^{max}), r_{out}(d_l^{min} \leftarrow d_l)]$$
$$+ min[\{r_{out}(d_l^{min} \rightarrow d_l^{max}) + r_{in}(d_l^c \leftarrow d_l^{max})\}, \{r_{out}(d_l^{min} \leftarrow d_l^{max}) + r_{in}(d_l^{min} \rightarrow d_l^c)\}].$$

$$R(d_l^{ss}) \ = \ \sum_{d_l \neq d_l^{min}, d_l^{ss}, d_l^{max}} min[r_{out}(d_l \rightarrow d_l^{max}), r_{out}(d_l^{min} \leftarrow d_l)]$$
$$+ min[\{r_{out}(d_l^{min} \rightarrow d_l^{max}) + r_{in}(d_l^{ss} \leftarrow d_l^{max})\}, \{r_{out}(d_l^{min} \leftarrow d_l^{max}) + r_{in}(d_l^{min} \rightarrow d_l^{ss})\}].$$

The second term in the both the above expressions will tend to zero for small δ. Thus,

$$R(d_l^c) - R(d_l^{ss}) \ = \ r_{out}(d_l^{min} \leftarrow d_l^{ss}) - min[r_{out}(d_l^c \rightarrow d_l^{max}), r_{out}(d_l^{min} \leftarrow d_l^c)].$$

$$\Rightarrow R(d_l^c) - R(d_l^{ss}) > 0, \quad \forall \ d_l^c \ (\neq d_l^{ss}) \in D_l,$$

as the equilibrium at d_l^{ss} is the most difficult to escape. Hence, the equilibrium at d_l^{ss} is the stochastically stable outcome of the dynamic process.

A.2. Proof of Proposition 2

In $G(cg)$, we showed that the least costly transition from the equilibrium at d_l into D_l^+ is the local transition. Equation (16) gives rise to the following inequalities which hold true.

$$
\begin{aligned}
d_l \cdot (e - d_l) &\geq \delta \cdot (e - \delta) && \text{for } d_l < e \leq c_l. \\
d_l \cdot (e - d_l) &\geq c_l \cdot \delta && \text{for } d_l < c_l < e \leq c_h. \\
c_l \cdot (e - d_l) &\geq c_l \cdot \delta && \text{for } c_l < d_l < e \leq c_h. \\
d_l \cdot c_h &\geq c_l \cdot 0 && \text{for } d_l \leq c_l < c_h < e. \\
c_l \cdot (e - d_l) &\geq c_l \cdot \delta && \text{for } c_l < d_l \leq c_h < e. \\
c_l \cdot (e - d_l) &\geq c_l \cdot \delta && \text{for } c_l < c_h < d_l < e.
\end{aligned}
$$

Hence,

$$
f[d_l \to min(d_l^+)] \;=\; \frac{min(d_h, c_h) - Max[min(d_h^-, c_h)]}{min(d_h, c_h)}, \quad \forall\, d_l \in D_l. \tag{31}
$$

$$
\Rightarrow \; r_{out}^+(d_l) \;=\; \begin{cases} \frac{\delta}{e-d_l} & \text{if } d_l \geq e - c_h. \\ 0 & \text{if } d_l < e - c_h. \end{cases} \tag{32}
$$

Note that the resistance to move from an equilibrium in which the high claimants are demanding more than their contribution is zero. This is because if even an insignificant fraction of low claimants increase their demand slightly, all high claimants will have the incentive to state the corresponding lower equilibrium demand. Now let us consider the transitions from d_l into D_l^-. Following the same notation,

$$f_l[d_l^- \leftarrow d_l] \;=\; \frac{min(d_h, c_h)}{min(d_h^+, c_h)}, \quad \forall\, d_l \in D_l. \tag{33}$$

$$f_h[d_l^- \leftarrow d_l] \;=\; \frac{min(d_l, c_l) - min(d_l^-, c_l)}{min(d_l, c_l)}, \quad \forall\, d_l \in D_l. \tag{34}$$

The least costly transition towards the left of d_l initiated by the experiments of low claimants is the extreme transition as minimizing $f_l[d_l^- \leftarrow d_l]$ requires maximizing $min(d_h^+, c_h)$, which in turn implies that d_l^- should be as low as possible. Similarly, the least costly transition towards the left of d_l initiated by the experiments of high claimants is the local transition as minimizing $f_h[d_l^- \leftarrow d_l]$ requires maximizing $min(d_l^-, c_h)$, which in turn implies that d_l^- should be as large as possible. The overall least costly transition towards left of any d_l is again the local transition $[max(d_l^-) \leftarrow d_l]$, but in this case it is initiated by the experiments of H claimants. The proof is exactly similar as for the least costly transition to the right, and hence omitted. The resistance is

$$f[max(d_l^-) \leftarrow d_l] \;=\; \frac{min(d_l, c_l) - Max[min(d_l^-, c_l)]}{min(d_l, c_l)}, \quad \forall\, d_l \in D_l. \tag{35}$$

$$\Rightarrow \;\; r_{out}^-(d_l) \;=\; \begin{cases} \frac{\delta}{d_l} & \text{if } \; d_l \le c_l. \\ 0 & \text{if } \; d_l > c_l. \end{cases} \tag{36}$$

The values of $r_{out}^+(d_l)$ and $r_{out}^-(d_l)$ suggest that we effectively need to consider equilibria with $d_l \in [max(e - c_h, 0), min(e, c_l)]$ since the equilibria at all the other equilibrium values of d_l will be trivially easy to escape. Note that $r_{out}^+(d_l)$ is monotonically increasing in d_l, and $r_{out}^-(d_l)$ is monotonically decreasing in d_l for $d_l \in [max(e - c_h, 0), min(e, c_l)]$. $r_{out}^+(d_l) \; < \; r_{out}^-(d_l)$ for all $d_l \in [max(e - c_h, 0), min(e, c_l)]$ if $e \; > \; 2c_l$. $r_{out}^+(d_l)$ intersects $r_{out}^-(d_l)$ at $\frac{1}{2}e$ if $e \; \le \; 2c_l$. The minimal tree is given by the lower envelope of $[r^{+}_{out}(d_l),\ r^{-}_{out}(d_l)]$. It is rooted at $d_l^{ss} = \frac{1}{2}e$ or c_l depending upon whether e is smaller or greater than $2c_l$. It involves connecting the node at any $d_l > d_l^{ss}$ to $(d_l - \delta)$ and the node at any $d_l < d_l^{ss}$ to $(d_l + \delta)$. Moreover, the equilibrium at d_l^{ss} is the most difficult to escape. Thus, the stochastically stable equilibrium exactly corresponds to the division suggested by the CEA rule.

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